DISCUSSION: QUANTUM MECHANICS AND VALUE DEFINITENESS*

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"A key to the Kochen and Specker proof," notes Allen Stairs,[1] "is the fact that there is no way of assigning the value zero to the spin component in exactly one out of every three mutually orthogonal directions, even in certain finite cases."

Thus if $s(x)$ is the value of the observable 'spin in the x direction' of a (massive) spin-1 particle the equation:

$$s^2(x) + s^2(y) + s^2(z) = 2 \tag{1}$$

cannot be satisfied for *every* orthogonal triple $\{x,y,z\}$. This equation, however, is satisfied by the quantum mechanical observable s. It follows that the observable 'spin in the x-direction' does not always have a definite value (corresponding with the result that would be observed had we chosen to measure the spin in the x direction).

There exists, however, a considerable gap between the logical argument of Kochen and Specker and the results of actual quantum mechanical measurements which are statistical in nature. To see that let s be a function from the set of directions in space that takes its values in the set $\{-1,0,1\}$. Call an orthogonal triple $\{x,y,z\}$ a *good* triple if equation (1) is satisfied, and call it a *bad* triple if $s^2(x) + s^2(y) + s^2(z) \neq 2$. If we can construct a function s for which the family of bad triples is small so that 'almost all' triples are good, we can recover the *statistical* results of quantum mechanics (since the chance of hitting a bad triple is null), and we can still maintain that the spin value is definite. In the following I shall give the details of a construction of such a function. I shall use the continuum hypothesis along the way, but the result also follows from a much weaker assumption (e.g., Martin's axiom).

Let S^2 be the set of all directions (i.e., unit vectors) in the three dimensional Euclidean space. For $x \in S^2$ let $c(x)$ denote the set of all directions orthogonal to x. $c(x)$ is a major circle on the sphere S^2. Using

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[1]See Stairs 1983.

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the axiom of choice we can well order the set of all major circles $\{c(x_\alpha)|\alpha < \Omega\}$ where Ω is the least ordinal whose cardinality is $2^{\aleph_0}$. Now define a function $s:S^2 \rightarrow \{-1,0,1\}$ by induction on the order.

First Step: Put $s(x_1) = s(-x_1) = 0$. Divide the circle $c(x_1)$ into two equal halves, $c^1(x_1)$ and $c^2(x_1)$ (Fig. 1), and define $s(w) = 1$ for $w \in c(x_1)$ and $s(w) = -1$ for $w \in c^2(x_1)$.

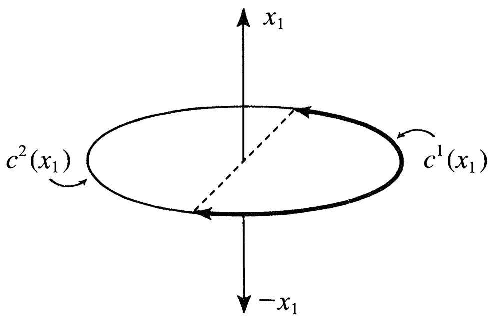

Figure 1

Let $\alpha < \Omega$. Suppose that we have defined the function s on all sets of the form $c(x_\beta) \cup \{-x_\beta, x_\beta\}$ for $\beta < \alpha$, we shall now define it on $c(x_\alpha) \cup \{-x_\alpha, x_\alpha\}$. If $x_\alpha = \pm x_\beta$ for some $\beta < \alpha$ the function s has already been defined. Otherwise consider the set $\cup_{\beta<\alpha}[c(x_\alpha) \cap c(x_\beta)]$. The intersection of any pair of nonidentical circles contains two opposite points and the cardinality of α is strictly less than $2^{\aleph_0}$. If we take the continuum hypothesis as valid it follows that $\cup_{\beta<\alpha}[c(x_\alpha) \cap c(x_\beta)]$ is countable. We have three cases:

(a) If $s(x_\alpha)$ has *not* been defined put $s(x_\alpha) = s(-x_\alpha) = 0$, divide the circle $c(x_\alpha)$ into two halves, $c^1(x_\alpha)$, $c^2(x_\alpha)$, and define

$$s(w) = 1 \qquad w \in c^1(x_\alpha) \setminus \bigcup_{\beta<\alpha} [c(x_\alpha) \cap c(x_\beta)]$$

$$s(w) = -1 \qquad w \in c^2(x_\alpha) \setminus \bigcup_{\beta<\alpha} [c(x_\alpha) \cap c(x_\beta)]$$

(b) If $s(x_\alpha)$ has been defined (i.e., $x_\alpha \in \cup_{\beta<\alpha} c(x_\beta)$), and $s(x_\alpha) = 0$ proceed as in case (a).

(c) If $s(x_\alpha)$ has been defined and $s(x_\alpha) = 1$ or $s(x_\alpha) = -1$, divide the circle $c(x_\alpha)$ into four equal quarters $c^1(x_\alpha)$, $c^2(x_\alpha)$, $c^3(x_\alpha)$, $c^4(x_\alpha)$ (Fig. 2). and define

$$s(w) = 0 \qquad w \in c^1(x_\alpha) \cup c^3(x_\alpha) \setminus \bigcup_{\beta<\alpha} [c(x_\alpha) \cap c(x_\beta)]$$

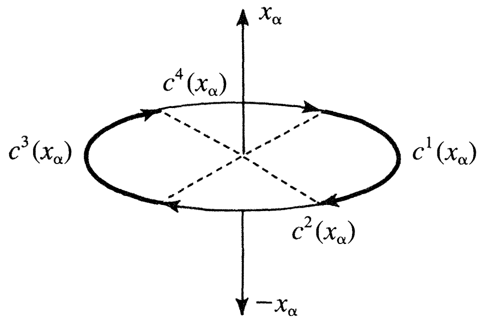

Figure 2

$$s(w) = 1 \qquad w \in c^2(x_\alpha) \setminus \bigcup_{\beta<\alpha} [c(x_\alpha) \cap c(x_\beta)]$$

$$s(w) = -1 \qquad w \in c^4(x_\alpha) \setminus \bigcup_{\beta<\alpha} [c(x_\alpha) \cap c(x_\beta)]$$

This completes the construction.

Now let $\{x,y,z\}$ be an arbitrary orthogonal triple. If $s(x) = 0$ then $s^2(w) = 1$ for all $w \in c(x)$ save perhaps countably many. Hence, $s^2(x) + s^2(y) + s^2(z) = 2$ for almost all pairs y, z orthogonal to x. If $s(x) = \pm 1$ then one, and only one, of the elements in the pair y, z lie in $c^1(x) \cup c^3(x)$; hence $s^2(x) + s^2(y)$ pl $s^2(z) = 2$ for all pairs y, z orthogonal to x save, perhaps, countably many. (These observations are, of course, independent of the order in which we consider the elements of the triple $\{x,y,z\}$.) Kochen and Specker's theorem has its limitations. Once we exchange the quantifier 'for all' by the weaker quantifier 'for almost all', *logical* difficulties disappear while the statistical results remain intact.[2]

REFERENCES

Pitowsky, Itamar (1983), "Deterministic Model of Spin and Statistics", *Physical Review D 27*: 2316–26.

Stairs, Allen (1983), "Quantum Logic Realism and Value Definiteness", *Philosophy of Science 50*: 578–602.

[2]For a detailed spin statistics model constructed along these lines, see my 1983.