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Local Fluctuations and Local Observers in Equilibrium Statistical Mechanics

*Itamar Pitowsky**

The distribution function associated with a classical gas at equilibrium is considered. We prove that apart from a factorisable multiplier, the distribution function is fully determined by the correlations among local momenta fluctuations. Using this result we discuss the conditions which enable idealised local observers, who are immersed in the gas and form a part of it, to determine the distribution 'from within'. This analysis sheds light on two views on thermodynamic equilibrium, the 'ergodic' and the 'thermodynamic limit' schools, and the relations between them. It also provides an outline for a new definition of equilibrium that is weaker than full ergodicity. Finally, we briefly discuss the possibility that the distribution can be determined by external observers. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Equilibrium Distribution; Fluctuations; Thermodynamic Limit; Ergodic Theory.

1. Introduction

For a classical gas, in a state of thermodynamic equilibrium molecules in all ranges of velocity are uniformly distributed in all parts of the container. However, we do expect that in any given macroscopic spatial region, the number of high speed (or low speed) molecules will occasionally increase above average for a brief moment. These small local momentum fluctuations, and the *correlations* among local momentum fluctuations at different regions, determine the distribution. In Section 3 we shall represent the distribution in

*Department of Philosophy, The Hebrew University, Mount Scopus, Jerusalem 91905, Israel (*e-mail:* itamarp@vms.huji.ac.il).

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terms of the correlations among local momentum fluctuations (Eqs (13) and (14)), and extend a previous result (Pitowsky and Shoresh, 1996).

This observation can be formulated in 'operational' language as follows: imagine small observers immersed in the gas (or liquid) and forming a part of it. If they are capable of detecting momentum fluctuations in their region, they can use the correlations among the fluctuations to communicate. In other words, they may be able to determine the distribution 'from within'. The necessary conditions for this to be possible are discussed in Section 4. The purpose of this exercise is to shed light on two approaches to thermodynamic equilibrium and the relationships between them, namely, the 'ergodic' and the 'thermodynamic limit' schools. We also suggest a new definition of equilibrium distribution based on our operational analysis.

Finally, in Section 5, we discuss the possibility that the distribution can be inferred from macroscopic fluctuation phenomena.

2. Definitions and Notations

Consider a classical gas of n particles with positions $x_1, x_2, \ldots, x_n$ and momenta $p_1, p_2, \ldots, p_n$. Let $\rho(x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n)$ be the distribution over phase space associated with it. To avoid the mathematical complications associated with the thermodynamic limit, and to make the discussion as general as possible, we shall not assume that ρ is the canonical Gibbs distribution. Rather, we shall make four assumptions:

1. The distribution $\rho(x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n)$ is integrable with respect to the Lebesgue measure on $\Re^{6n}$.
2. The distribution ρ is stationary.
3. ρ is symmetric with respect to the interchange of particles. In other words:

$$\rho(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}, p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(n)}) = \rho(x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n), \quad (1)$$

   for every permutation π of $1, 2, \ldots, n$.
4. For each n the corresponding distribution ρ exists and is obtained from the higher-dimensional distributions as marginal. In other words, Kolmogorov's consistency conditions obtain (Chow and Teicher, 1978, p. 185).

Given a distribution ρ over n particles we can define its marginals:

$$\rho^{(1)}(x_1, p_1) = \int\int \cdots \int \rho(x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n) d^3x_2 d^3x_3 \ldots d^3x_n d^3p_2 d^3p_3 \ldots d^3p_n,$$

$$\rho^{(2)}(x_1, x_2, p_1, p_2) = \int\int \cdots \int \rho(x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n) d^3x_3 d^3x_4 \ldots d^3x_n d^3p_3 d^3p_4 \ldots d^3p_n, \quad (2)$$

and so forth, for the higher-dimensional marginals. Here the integration ranges over the entire space and all possible momenta values. Because of the symmetry condition (1) we know that the single particle marginal distribution for particle number i is just $\rho^{(1)}(x_i, p_i)$, the two particle marginal for particles i, j is $\rho^{(2)}(x_i, x_j, p_i, p_j)$ and so on.

Let $f(x, p)$ be a real function. We shall consider thermodynamic observables of the form:

$$F(x_1, x_2, \dots, x_n, p_1, p_2, \dots, p_n) = \frac{1}{n}\sum_{i=1}^{n} f(x_i, p_i). \tag{3}$$

Let A be a small region in physical space: $A = \{x : \|x - x_0\| < \varepsilon\}$ where $x_0 \in \Re^3$ is fixed and $\varepsilon > 0$ is small. Denote by $\chi_A(x)$ the characteristic function of A, which equals 1 when $x \in A$ and 0 otherwise. Put

$$F_A(x_1, x_2, \dots, x_n, p_1, p_2, \dots, p_n) = \frac{1}{n}\sum_{i=1}^{n} f(x_i, p_i)\chi_A(x_i). \tag{4}$$

F_A is then the restriction of the observable F to the region A.

If $A_1, A_2, \dots, A_k$, with $k \ll n$, are small disjoint regions in physical space, then the k-fold correlation between the local fluctuations of F in the regions $A_1, A_2, \dots, A_k$ is given by:

$$\begin{aligned}&\langle (F_{A_1} - \langle F_{A_1}\rangle)(F_{A_2} - \langle F_{A_2}\rangle)\cdots(F_{A_k} - \langle F_{A_k}\rangle)\rangle \\ &\quad = \sum_{\phi \subseteq \beta \subseteq \{1,2,\dots,k\}} (-1)^{k-|\beta|}\langle \prod_{i\in\beta} F_{A_i}\rangle \prod_{j\notin\beta}\langle F_{A_j}\rangle,\end{aligned} \tag{5}$$

where $|\beta|$ is the number of elements in $\beta \subseteq \{1, 2, \dots, k\}$. We denote

$$\begin{aligned}\alpha^{(m)}(f, A_{i_1}, A_{i_2}, \dots, A_{i_m}) &= \int_{-\infty}^{\infty} d^3p_{i_1}\cdots\int_{-\infty}^{\infty} d^3p_{i_m}\int_{A_{i_1}} d^3x_{i_1}\cdots\int_{A_{i_m}} d^3x_{i_m} \\ f(x_{i_1}, p_{i_1}) \times \cdots \times f(x_{i_m}, p_{i_m})&\rho^{(m)}(x_{i_1}, \dots, x_{i_m}, p_{i_1}, \dots, p_{i_m}),\end{aligned} \tag{6}$$

where $\rho^{(m)}$ is the marginal for m particles. Since the A_i's are pairwise disjoint, the correlation (5) is given by:

$$\begin{aligned}&\langle (F_{A_1} - \langle F_{A_1}\rangle)(F_{A_2} - \langle F_{A_2}\rangle)\cdots(F_{A_k} - \langle F_{A_k}\rangle)\rangle = \\ &\sum_{\phi\subseteq\{i_1,\dots,i_m\}\subseteq\{1,\dots,k\}} (-1)^{k-m}\frac{n!}{n^m(n-m)!}\alpha^{(m)}(f, A_{i_1}, A_{i_2}, \dots, A_{i_m}) \\ &\times \prod_{j\notin\{i_1,\dots,i_m\}} \alpha^{(1)}(f, A_j).\end{aligned} \tag{7}$$

Here $\alpha^{(0)} = 1$. Taking the limit $n \to \infty$, we obtain:

$$\begin{aligned}&\lim_{n\to\infty} \langle (F_{A_1} - \langle F_{A_1}\rangle)(F_{A_2} - \langle F_{A_2}\rangle)\cdots(F_{A_k} - \langle F_{A_k}\rangle)\rangle = \\ &\sum_{\phi\subseteq\{i_1,\dots,i_m\}\subseteq\{1,\dots,k\}} (-1)^{k-m}\alpha^{(m)}(f, A_{i_1}, A_{i_2}, \dots, A_{i_m}) \prod_{j\notin\{i_1,\dots,i_m\}} \alpha^{(1)}(f, A_j).\end{aligned} \tag{8}$$

We see that the correlations between local fluctuations of a thermodynamic observable do not necessarily diminish in the limit $n \to \infty$.

3. Correlations of Local Momentum Fluctuations

Of particular interest is the case where F is the average number of particles that have a given range of momenta. To be more precise, assume that p_0 is a certain value of momentum (p_0 is a three-dimensional vector). Let $B = \{p : \|p - p_0\| < \varepsilon\}$ be a small neighbourhood of p_0, where $\varepsilon > 0$ is small. Set $f = \chi_B(p)$, the characteristic function of B, then $F = \frac{1}{n}\sum_{i=1}^{n} \chi_B(p_i)$ is the proportion of particles with momenta in B. If A is a small region in physical space then $F_A = \frac{1}{n}\sum_{i=1}^{n} \chi_B(p_i)\chi_A(x_i)$ is the proportion of particles in A with momenta in B.

More generally, if $A_1, A_2, \ldots, A_k$ are *disjoint*, small regions in space and $B_1, B_2, \ldots, B_k$ are k neighbourhoods of momenta values, we set $f(x,p) = \sum_{j=1}^{k} \chi_{B_j}(p)\chi_{Aj}(x)$. Then $F_{A_j} = n^{-1}\sum_{i=1}^{n} \chi_{B_j}(p_i)\chi_{Aj}(x_i)$. Thus, in this case, formula (8) gives the k-fold correlations among the fluctuations in the number of particles in region A_j, with range of momenta in B_j, $j = 1, 2, \ldots k$ (in the limit $n \to \infty$). Denote this value by $C^{(k)}(A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k)$. By direct calculation we get ($\alpha^{(0)} = 1$):

$$C^{(k)}(A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k) = \sum_{\phi \subseteq \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, k\}} (-1)^{k-m} \alpha^{(m)}(A_{i_1}, A_{i_2}, \ldots, A_{i_m}, B_{i_1}, B_{i_2}, \ldots, B_{i_m}) \times \prod_{j \notin \{i_1, \ldots, i_m\}} \alpha^{(1)}(A_j, B_j), \tag{9}$$

where

$$\alpha^{(m)}(A_{i_1}, A_{i_2}, \ldots, A_{i_m}, B_{i_1}, B_{i_2}, \ldots, B_{i_m}) = \int_{B_{i_1}} \cdots \int_{B_{i_m}} \int_{A_{i_1}} \cdots \int_{A_{i_m}} \rho^{(m)}(x_{i_1}, \ldots, x_{i_m}, p_{i_1}, \ldots, p_{i_m}) \times d^3x_{i_1} \ldots d^3x_{i_m} d^3p_{i_1} \ldots d^3p_{i_m}. \tag{10}$$

In particular, $C^{(2)}(A_1, A_2, B_1, B_2)$ is the correlation between the fluctuations in the (average) number of particles in A_1 with momentum in B_1, and the fluctuations in the average number of particles in A_2 having momentum in B_2. All that, of course, after taking the limit $n \to \infty$. If, for example, $C^{(2)}(A_1, A_2, B_1, B_2) < 0$, and the number of particles in A_1 with momentum in B_1 fluctuates above average then, with a high probability, the number of particles in A_2 having momentum in B_2 is below average.

Instead of the correlations $C^{(k)}$ it is more convenient to define the *correlation density*. Assume that each A_i is a small neighbourhood of some x_i^0, that is, $A_i = \{x : \|x - x_i^0\| < \varepsilon\}$ and similarly, each B_i is a small neighbourhood of a fixed p_i^0; then, since ρ and its marginals $\rho^{(m)}$ are Lebesgue integrable we get:

$$\lim_{\varepsilon\downarrow 0}\frac{1}{(4\pi\varepsilon^3)^{2m}}\int_{B_{i_1}}\cdots\int_{B_{i_m}}\int_{A_{i_1}}\cdots\int_{A_{i_m}}\rho^{(m)}(x_{i_1},\ \ldots,x_{i_m},p_{i_1},\ \ldots,p_{i_m})$$
$$\times\, d^3x_{i_1}\cdots d^3x_{i_m}d^3p_{i_1}\cdots d^3p_{i_m}=\rho^{(m)}(x^0_{i_1},\ \ldots,x^0_{i_m},p^0_{i_1},\ \ldots,p^0_{i_m}), \tag{11}$$

for almost every choice of x^0_i, and p^0_i. By (10) and (11) we obtain that the correlation density

$$c^{(k)}(x^0_1,x^0_2,\ \ldots,x^0_k,p^0_1,p^0_2,\ \ldots,p^0_k)=$$
$$\lim_{\varepsilon\downarrow 0}\frac{1}{(4\pi\varepsilon^3)^{2k}}C^{(k)}(A_1,A_2,\ \ldots,A_k,B_1,B_2,\ \ldots,B_k)=$$
$$\sum_{\phi\subseteq\{i_1,\ \ldots,i_m\}\subseteq\{1,\ \ldots,k\}}(-1)^{k-m}\rho^{(m)}(x^0_{i_1},x^0_{i_2},\ \ldots,x^0_{i_m},p^0_{i_1},p^0_{i_2},\ \ldots,p^0_{i_m})$$
$$\times\prod_{j\notin\{i_1,\ \ldots,i_m\}}\rho^{(1)}(x^0_j,p^0_j) \tag{12}$$

is almost everywhere well defined (here $\rho^{(0)}=1$).

If we put $c^{(0)}\equiv 1$, $c^{(1)}(x,p)\equiv 0$, we can invert (12) and express the distribution ρ in terms of the correlation densities $c^{(k)}$ and the first marginal $\rho^{(1)}$:

$$\rho(x_1,x_2,\ \ldots,x_n,p_1,p_2,\ \ldots,p_n)=$$
$$\sum_{\phi\subseteq\{i_1,\ \ldots,i_m\}\subseteq\{1,\ \ldots,n\}}c^{(m)}(x_{i_1},x_{i_2},\ \ldots,x_{i_m},p_{i_1},p_{i_2},\ \ldots,p_{i_m})$$
$$\times\prod_{j\notin\{i_1,\ \ldots,i_m\}}\rho^{(1)}(x_j,p_j). \tag{13}$$

Or, in another form,

$$\rho(x_1,x_2,\ \ldots,x_n,p_1,p_2,\ \ldots,p_n)$$
$$=\prod_{j=1}^{n}\rho^{(1)}(x_j,p_j)\left(1+\sum_{0\leqslant i<j\leqslant n}\frac{c^{(2)}(x_i,x_j,p_i,p_j)}{\rho^{(1)}(x_i,p_i)\rho^{(1)}(x_j,p_j)}\right.$$
$$\left.+\sum_{0\leqslant i<j<k\leqslant n}\frac{c^{(3)}(x_i,x_j,x_k,p_i,p_j,p_k)}{\rho^{(1)}(x_i,p_i)\rho^{(1)}(x_j,p_j)\rho^{(1)}(x_k,p_k)}+\ldots\right). \tag{14}$$

Formulas such as (14) have been derived previously in a somewhat more restricted context, in particular in the theory of the liquid state (for example Hill, 1956, p. 183). What is novel here is the demonstration that the functions $c^{(k)}$ are the limit $n\to\infty$ of the correlation densities among local momenta fluctuations.

An immediate consequence of (14) states the following: a necessary and sufficient condition that the distribution function ρ is *factorisable*

$$\rho(x_1,x_2,\ \ldots,x_n,p_1,p_2,\ \ldots,p_n)=\prod_{j=1}^{n}\rho^{(1)}(x_j,p_j) \tag{15}$$

is that the correlations among local momenta fluctuations $c^{(k)}$ diminish, for $k = 1, 2, \ldots$. Now, add the assumption that the distribution is a function of the Hamiltonian H, of the form

$$H = \sum \frac{p_i^2}{2m} + V(x_1, x_2, \ldots, x_n),$$

where V is the potential of the interparticle forces. In this case we obtain:

> *A necessary and sufficient condition that the distribution has the (Maxwellian) form*
>
> $$\rho(x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n) = \frac{1}{Z} \exp\left(- \frac{1}{kT} \sum_{i=1}^{n} \frac{p_i^2}{2m} \right) \tag{16}$$
>
> *is that the correlations among local momentum fluctuations diminish in the limit $n \to \infty$* (Pitowsky and Shoresh, 1996).

4. Local Observers: Ergodicity and Equilibrium

If the fluctuation correlation densities $c^{(k)}$ are known, we can derive the distribution ρ from (14). Thus, we infer the form of the distribution (16) from the assumption that there are no correlations among local momenta fluctuations. Not surprisingly, perhaps, this turns out to be equivalent to the lack of forces acting between the particles (that is, the gas is ideal). Indeed, formula (16) is often derived after (15) is simply *assumed* to be valid for an ideal gas (see, for example, Landau and Lifshitz (1980)).

But we can ask a deeper question: can we *define* the distribution ρ in terms of Eqs (13) and (14)? In other words, is there a way to define the correlation densities $c^{(k)}$ independently of the probability distribution ρ? If this can be done then, by (13) and (14), we can define the distribution up to the normalisation factors $\prod_j \rho^{(1)}(x_j, p_j)$.

One possible way to approach the question is to ask whether the $c^{(k)}$'s can somehow be observed from outside the system. Since local fluctuations give rise to macroscopically observable phenomena, this question is not entirely far-fetched, and we shall comment on it in the next section. For the moment we shall adopt a more idealised approach and consider the question whether local observers, who are immersed in the gas and form a part of it, can determine ρ.

The approach that we propose is similar to the operational view of spacetime theories. In the latter, the metric structure of spacetime is operationally determined by idealised local observers equipped with clocks and rulers. The observers perform spacetime measurements in their respective locations, and 'compare notes' using electromagnetic communication. However, it would be a mistake to think that one can operationally define the metric structure of spacetime without circularity (Putnam, 1975). What is achieved, though, is an *analysis* of the assumptions involved in the introduction of the metric structure, and the relations of these assumptions to observation. By analogy, we cannot expect our analysis to eliminate the concept of probability from statistical mechanics. The aim, rather, is to shed light on the concept of equilibrium and the role of time averages in statistical mechanics.

One notorious local 'observer' whose business is to measure local momentum fluctuations is Maxwell's demon. However, there is a crucial difference between our observers and the demon in that our observers do not necessarily change the entropy of the system. All they are capable of is observing the velocity of the particles flying around them, store in memory the data concerning past observations, and perform simple calculations.[1]

To illustrate the approach consider the magnitude $C^{(2)}(A_1, A_2, B_1, B_2)$, the correlation between the fluctuations in the (average) number of particles in A_1 with momentum values in B_1, and the fluctuations in the number of particles in A_2 having momentum in B_2. Assume that two local observers, call them Alice and Bob, are located in regions A_1 and A_2 respectively. Each of these regions is enclosed with solid walls and a gate with a door through which particles can come and go. The regions have small but macroscopic dimensions, and are connected to one another by a pipe with doors at its ends (Fig. 1).

Assume, for simplicity, that the momenta are isotropically distributed, so that ρ depends on p_i^2, and not on the direction of p_i. Let us in this case consider the sets B_j, $j = 1, 2$ to have the form $B_j = \{p_i : a_j - \varepsilon < p_i^2 < a_j + \varepsilon\}$, where $a_j > 0$ are real numbers. To prepare their system Alice and Bob leave the doors connecting their cells to the outside world open, and the doors to the pipe connecting their cells closed. Now, suppose that at a certain moment Alice senses that the number of particles with the momentum range $a_1 - \varepsilon < p_i^2 < a_1 + \varepsilon$ increases above average or decreases below average. She immediately closes the door to the outside world. Bob does the same if he senses a change above or below in the average number of particles with $a_2 - \varepsilon < p_i^2 < a_2 + \varepsilon$. Suppose that $C^{(2)}(A_1, A_2, B_1, B_2) < 0$. In this case it is probable that an above average count in Alice's cell will correspond to a simultaneous below average count in Bob's cell and vice versa. Suppose, for example, that Alice's count is above average and Bob's count is below average. In this case, upon opening the doors to the pipe, the densities of particles of the kind with $p_i^2 \in (a_1 - \varepsilon, a_1 + \varepsilon)$ and the kind with $p_i^2 \in (a_2 - \varepsilon, a_2 + \varepsilon)$ will become more uniform. Therefore, Alice will observe particles of *both* kinds leaving her cell, and Bob will see particles of both kinds arrive at his cell. The number of such particles, which is known to both Alice and Bob, indicates the strength of

[1] Some authors think that the ability to store and manipulate information necessarily implies an increase of entropy. On this and other (circular) 'proofs' that Maxwell's demon is physically impossible see Earman and Norton (1998, 1999). (But see also Shenker (1999) for a different impossibility argument that does not pertain to the present paper.)

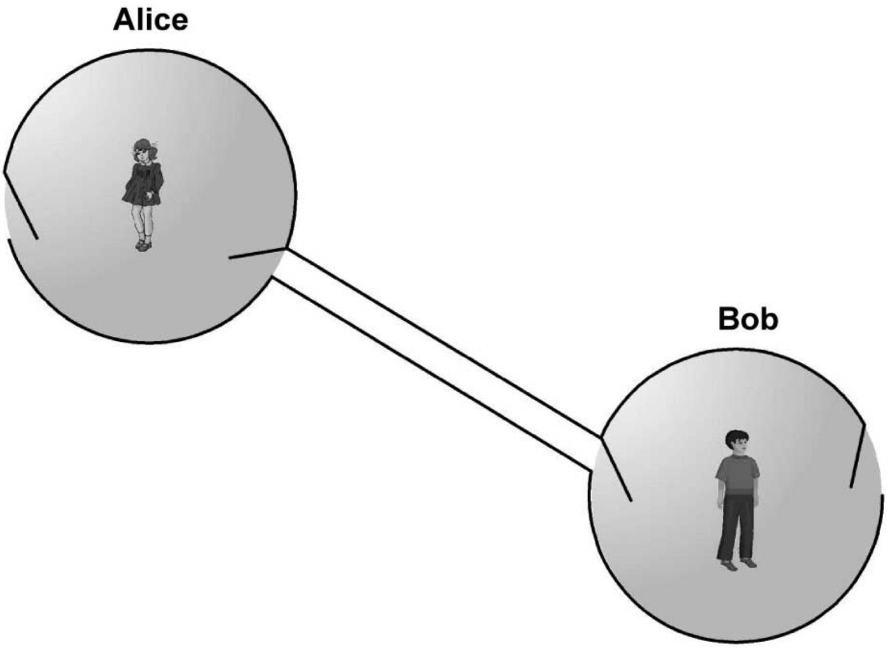

Fig. 1. *Two local observers, located in regions of small but macroscopic dimensions, and connected by a pipe.*

correlation. Since we do not want our observers to change the distribution (and entropy), we assume that the entire affair is very swift, and takes no longer than it takes the fluctuation to recede normally. Similarly, if $C^{(2)}(A_1, A_2, B_1, B_2) > 0$, and Alice's count is above average and so is Bob's, particles of one kind, viz $p_i^2 \in (a_1 - \varepsilon, a_1 + \varepsilon)$ will seem to flow from Alice to Bob, and of the other kind, viz $p_i^2 \in (a_2 - \varepsilon, a_2 + \varepsilon)$ from Bob to Alice. Note that in order to 'compare notes' the observers need no external means of communication. If the fluctuation correlations are non-zero it provides for the information flow. If it is zero no information exchange is needed.

A similar analysis applies in the case of three or more local observers (Fig. 2).

When we say that the observers measure averages we mean *time averages*. Thus, for example, when Alice measures the proportion of particles in A_1, with momenta in the range B_1, she actually averages that proportion over time.

Now, Alice's measurements of this average will coincide with the external probability measure ρ if, for sufficiently long τ, this time average converges to the marginal measure of $A_1 \times B_1$, that is

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \left(\frac{1}{n} \sum_{i=1}^{n} \chi_{B_1}(p_i(t)) \chi_{A_1}(x_i(t)) \right) \mathrm{d}t = \int_{B_1} \int_{A_1} \rho^{(1)}(x, p) d^3x d^3p. \quad (17)$$

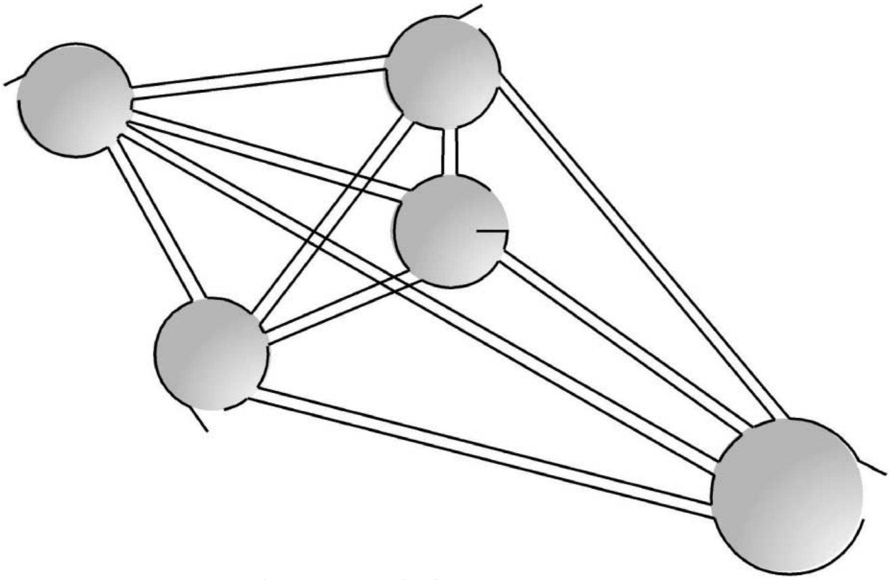

Fig. 2. *A network of several local observers.*

Note that this condition is weaker than the ergodic condition on the measure ρ.[2] The latter implies that *for each particle* $i = 1, 2, \ldots, n$ *alone* we shall have

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \chi_{B_1}(p_i(t))\chi_{A_1}(x_i(t))\mathrm{d}t = \int_{B_1}\int_{A_1} \rho^{(1)}(x_i, p_i)d^3x_i d^3p_i. \tag{18}$$

Moreover, when Alice and Bob measure the correlation they actually take the time average

$$\tau^{-1}\int n^{-2}\sum_{ij} \chi_{B_1}(i)\chi_{B_2}(j)\chi_{A_1}(i)\chi_{A_2}(j)\mathrm{d}t$$
$$-\left(\tau^{-1}\int (n^{-1}\sum_i \chi_{A_1}(i)\chi_{B_1}(j))\mathrm{d}t\right)\left(\tau^{-1}\int (n^{-1}\sum_j \chi_{A_2}(i)\chi_{B_2}(j))\mathrm{d}t\right). \tag{19}$$

We require that it converges, as $\tau \to \infty$, to $C^{(2)}(A_1, A_2, B_1, B_2)$. The ergodic condition for ρ is stronger, and implies that for each pair of particles ij alone

$$\tau^{-1}\int \chi_{B_1}(i)\chi_{B_2}(j)\chi_{A_1}(i)\chi_{A_2}(j)\mathrm{d}t$$
$$-\left(\tau^{-1}\int \chi_{A_1}(i)\chi_{B_1}(j)\mathrm{d}t\right)\left(\tau^{-1}\int \chi_{A_2}(i)\chi_{B_2}(j)\mathrm{d}t\right) \tag{20}$$

converges to $C^{(2)}(A_1, A_2, B_1, B_2)$ as $\tau \to \infty$.

[2] Even if we take ρ to be the limit measure $\lim_{n\to\infty} \rho^{(n)}$. This limit measure on $\Re^\infty$ exists by Kolmogorov's theorem (Chow and Teicher, 1978, p. 186). Regarding the ergodicity of measures like ρ, see the discussion below.

It is reasonable to assume that in normal circumstances, far enough from a phase transition, only very few lower correlation functions $c^{(k)}$ make an effective contribution to (14). This means, perhaps, that less than full ergodicity is required in order for the local observers to be able to infer the correct distribution. All this is in line with the approach to statistical mechanics due to Khinchin:

> All the results obtained by Birkhoff and his followers [...] pertain to the most general type of dynamic systems [...]. [They pertain] equally to the systems with only a few degrees of freedom as well as to the systems with a very large number of degrees of freedom.
>
> From our point of view we must deviate from this tendency. We would unnecessarily restrict ourselves by neglecting the special properties of the systems considered in statistical mechanics (first of all their fundamental property of having a very large number of degrees of freedom) [...]. Furthermore, we do not have any basis for demanding the possibility of substituting phase averages for the time averages of *all* functions; in fact the functions for which such substitution is desirable have many specific properties which make such a substitution apparent in these cases (Khinchin, 1949, p. 62).

What Khinchin suggests, roughly, is to replace the ergodic theorem by the law of large numbers. He shows (*ibid*., chapter VII) how this can be done with 'sum functions' of the form (3). The limit (17) represents such a case. Modern developments in the theory of the thermodynamic limit have generalised his approach considerably.

All this means that our local observers will be able to infer the distribution ρ from their time averages even under conditions less stringent then ergodicity. (Provided that the number of significant $c^{(k)}$'s in (14) is small compared to n.) If the measure ρ is, indeed, non-ergodic the local observers may err in their estimations of the expectations of some global properties of the gas. The phase space average of these global functions may differ from the time average, in opposition to the ergodic case. Note that our definition combines features taken from the ergodic program and the thermodynamic limit approach, not unlike Malament and Zabell (1980).

We can use all this to define equilibrium in a way that extends ergodicity. We shall say that *ρ represents equilibrium if local observers can sufficiently approximate ρ from time averages* in the manner described above. In other words *ρ is an equilibrium distribution if time averages and space averages of the $c^{(k)}$, for all significant k, coincide*. To make the definition precise one has to specify the degree of the required approximation.

One advantage of this definition over ergodicity is that non-ergodic measures, such as the one for an ideal gas, represent equilibrium even for finite n. Another advantage is that the definition may apply to systems that are not isolated. Take, for example, a gas in a container which is almost isolated except that its molecules interact gravitationally with distant external bodies. From classical mechanics we can infer that the total system—gas and external bodies—is likely to be radically non-ergodic (by the KAM theorem). If we concentrate on the gas itself we see that its trajectory, in its own phase space, is not confined to a single energy hypersurface, but fluctuates around the mean surface due to gravitational energy exchanges. The distribution function for the gas is thus non-singular. Yet, it very likely does represent an equilibrium state in our sense.[3]

Another case, not unlike the former, is that of the canonical distribution. Ever since the famous review (Ehrenfest and Ehrenfest, 1912), Boltzmann's point of view has been widely taken as conceptually superior to that of Gibbs.[4] However, it is the canonical (or grand canonical) and not the microcanonical distribution which is being used in calculations. This schizophrenic attitude is commonly justified by the appeal to the thermodynamic limit: 'All the ensembles are thermodynamically equivalent because as long as we are dealing with systems containing large numbers of molecules the most probable distribution is so overwhelmingly more probable than any other that it is virtually the average distribution' (Hecht, 1990, p. 28). Now, if it is the thermodynamic limit that does the job what is the superior conceptual role of ergodicity?

From our point of view one can see why the canonical distribution is typically an equilibrium distribution. If we think of it as representing a system in contact with a large heat bath, then, like in the gravitational case, most of the measure ρ is supported in a small neighbourhood of the mean energy surface. With n large enough we are likely to be able to show that ρ can be recovered by local observers, that is, it is an equilibrium distribution in our sense.

There are formal results that indicate that this is, in fact, the case. The modern theory of the thermodynamic limit[5] establishes the existence and properties of limit distributions in a variety of cases (Ruelle, 1969; Milnos, 2000). All ensembles often give the same limit distribution, the limit distribution is often ergodic. Also, in the limit distribution, correlations like $C^{(2)}(A_1, A_2, B_1, B_2)$ tend fast to 0 as the distance between the regions A_1 and A_2 grow. All this indicates that in many cases, even for finite n, local observers may be able to recover the distribution, that is, show that ρ is an equilibrium distribution.[6]

[3] This observation is related to the interventionist approach to statistical mechanics; about which see Hemmo and Shenker (2001).

[4] For example Pauli (1973). But Schrödinger (1946) knew better.

[5] We have avoided the complications of the thermodynamic limit by assuming the Kolmogorov consistency conditions (condition 4 in Section 2), and by identifying the finite-dimensional distributions with the marginals of the limit distribution. The thermodynamic limit is a more complicated concept (Ruelle, 1969; Milnos, 2000). However, if the thermodynamic limit distribution ρ^{ω} exists, then its marginals satisfy the Kolmogorov consistency conditions, which are *necessary* as well as sufficient. Hence, in particular, our limit $n \to \infty$ in (8) coincides with the thermodynamic limit. Therefore, our analysis still follows if we replace the simple $n \to \infty$ with the 'thermodynamic limit'.

[6] It is interesting to try and establish the relationships between the present concept of equilibrium (recoverability of ρ by local observers) and the one used in the theory of the thermodynamic limit, due to Dobrushin, Lanford and Ruelle (DLR) (Ruelle, 1969).

5. Can we see the Distribution?

From (14) it is clear that apart from the multiplicative factor $\prod_{j=1}^{n} \rho^{(1)}(x_j, p_j)$ that represents the 'ideal gas' case, the distribution depends on the correlations densities $c^{(k)}$. The question is, therefore, whether these correlations have macroscopic observable consequences, even far from a phase transition. In order to explore this possibility we have to look at macroscopically observable fluctuation phenomena such as Brownian motion. Unfortunately, I do not have a detailed theory, but only a few tentative proposals to make.

Consider a Brownian particle located at time t_0 at the position X_0. (For the sake of simplicity we shall use a model with a one-dimensional physical space.) The probability density that at time t_1 the particle will be position X is given by

$$\mathrm{d}P(X) = \frac{1}{\sqrt{2\pi D(t_1 - t_0)}} \exp\left[-\frac{(X - X_0)^2}{2D(t_1 - t_0)} \right] \mathrm{d}X, \tag{21}$$

where D, the diffusion coefficient, is given by Einstein's formula (Hecht, 1990, p. 348). Now, suppose that there are two Brownian particles located respectively at X_0 and Y_0 at time t_0. The probability density of finding the particles at X and Y at time t_1 is not necessarily simply the product of the single particle densities. In general it may have the form

$$\mathrm{d}P(X, Y) = M \exp\left[-\frac{(X - X_0)^2}{2D(t_1 - t_0)} - \frac{(Y - Y_0)^2}{2D(t_1 - t_0)} + \phi(X, Y) \right] \mathrm{d}X\mathrm{d}Y, \tag{22}$$

where $\phi(X, Y)$ expresses the correlation between the Brownian particles' positions and is assumed to be independent of time, and M is a normalisation factor (which does depend on time). Assuming that $\phi \neq 0$, there are two questions:

1. How can we observe the effects of ϕ?
2. What are the relations between ϕ and the correlation densities $c^{(k)}$ of the liquid in which the particles are suspended?

The first question is easy to answer, at least in principle. If ϕ differs from 0 significantly we can see a measurable correlation between the length of the paths of the two particles. For a sufficiently short $t_1 - t_0$ we shall observe

$$\int \int (X - X_0)(Y - Y_0)\mathrm{d}P(X, Y) \neq 0. \tag{23}$$

As for the second question, we can only give a qualitative answer. Let A_1 and A_2 be two disjoint intervals on the real line and let B be a range of momentum corresponding to large momentum values $B = \{p : |p| > \frac{2}{n}\sum|p_i|\}$. If, for example, $C^{(2)}(A_1, A_2, B, B) > 0$ then an increase in the number of particles with large momentum present at A_1 will correspond, with high probability, to an increase in the number of particles with large momentum in A_2. Assume that $X_0 \in A_1$ and $Y_0 \in A_2$. This means that as long as the two Brownian particles remain in A_1 and A_2 respectively, the length of the paths taken by the two particles at a short time $t_1 - t_0$ will be positively correlated (with high probability, both increase and decrease together). I do not know whether the correlations are strong enough, and last long enough, to produce an observable effect at the available level of resolution.

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