PERGAMON

Studies in History and Philosophy of Modern Physics 34 (2003) 395–414

Studies in History and Philosophy of Modern Physics

www.elsevier.com/locate/shpsb

Betting on the outcomes of measurements: a Bayesian theory of quantum probability

Itamar Pitowsky

Department of Philosophy, the Hebrew University, Mount Scopus, Jerusalem 91905, Israel

Abstract

We develop a systematic approach to quantum probability as a theory of rational betting in quantum gambles. In these games of chance, the agent is betting in advance on the outcomes of several (finitely many) incompatible measurements. One of the measurements is subsequently chosen and performed and the money placed on the other measurements is returned to the agent. We show how the rules of rational betting imply all the interesting features of quantum probability, even in such finite gambles. These include the uncertainty principle and the violation of Bell's inequality among others. Quantum gambles are closely related to quantum logic and provide a new semantics for it. We conclude with a philosophical discussion on the interpretation of quantum mechanics.

Keywords: Quantum gambling; Bayesian theory; Quantum probability; Bell's inequality; Quantum logic

1. Quantum gambles

1.1. The gamble

The Bayesian approach takes probability to be a measure of ignorance, reflecting our state of knowledge and not merely the state of the world. It follows Ramsey's contention that "we have the authority both of ordinary language and of many great thinkers for discussing under the heading of probability … the logic of partial belief" (Ramsey, 1926, p. 55). Here we shall assume, furthermore, that probabilistic beliefs are expressed in rational betting behavior: "The old-established way of measuring a person's belief … by proposing a bet, and see what are the lowest odds which he will

E-mail address: itamarp@vms.huji.ac.il (I. Pitowsky).

1355-2198/03/$ - see front matter
doi:10.1016/S1355-2198(03)00035-2

accept, is fundamentally sound."[1] My aim is to provide an account of the peculiarities of quantum probability in this framework. The approach is intimately related to and inspired by the foundational work on quantum information by Barnum, Caves, Finkelstein, Fuchs, and Schack (2000), Fuchs (2001), Schack, Brun, and Caves (2001) and Caves, Fuchs, and Schack (2002).

For the purpose of analyzing quantum probability, we shall consider *quantum gambles*. Each quantum gamble has four stages:

1. A *single* physical system is prepared by the bookie.
2. A *finite* set $\mathscr{M}$ of incompatible measurements is announced by the bookie, and the agent is asked to place bets on possible outcomes of each one of them.
3. One of the measurements in the set $\mathscr{M}$ is chosen by the bookie and the money placed on all other measurements is promptly returned to the agent.
4. The chosen measurement is performed and the agent gains or loses in accordance with his bet on that measurement.

We do not assume that the agent who participates in the game knows quantum theory. We do assume that after the second stage, when the set of measurements is announced, the agent is aware of the possible outcomes of each one of the measurements, and also of the relations (if any) between the outcomes of different measurements in the set $\mathscr{M}$. Let me make these assumptions precise. For the sake of simplicity we shall consider only measurements with a finite set of possible outcomes. Let A be an observable with n possible distinct outcomes $a_1, a_2, \ldots, a_n$. Each outcome corresponds to an *event* $E_i = \{A = a_i\}$, $i = 1, 2, \ldots, n$, and these events generate a Boolean algebra which we shall denote by $\mathscr{B} = \langle E_1, E_2, \ldots, E_n \rangle$. Subsequently, we shall identify the observable A with this Boolean algebra. Note that this is an unusual identification. It means that we equate the observables A and $f(A)$, whenever f is a one–one function defined on the eigenvalues of A. This step is justified since we are interested in *outcomes* and not their labels, hence the scale free concept of observable. With this, $\mathscr{M}$ is a finite family of finite Boolean algebras. Our first assumption is that the agent knows the number of possible distinct outcomes of each measurement in the set $\mathscr{M}$.

Our next assumption concerns the case where two measurements in the set $\mathscr{M}$ share some possible elements. For example, let A, B, C be three observables such that $[A, B] = 0, [B, C] = 0$, but $[A, C] \neq 0$. Consider the two incompatible measurements, the first of A and B together and the second of B and C together. If $\mathscr{B}_1$ is the Boolean algebra generated by the outcomes of the first measurement and $\mathscr{B}_2$ of the second, then $\mathscr{M} = \{\mathscr{B}_1, \mathscr{B}_2\}$ and the events $\{B = b_i\}$ are elements of both algebras; that is, of $\mathscr{B}_1 \bigcap \mathscr{B}_2$. We assume that the agent is aware of these facts when he is placing his bets.

[1] Ramsey (1926, p. 68). This simple scheme suffers from various weakness and better ways to associate epistemic probabilities with gambling have been developed (de Finetti, 1972). Any one of de Finetti's schemes can serve our purpose. For a more sophisticated way to associate probability and utility see Savage (1954).

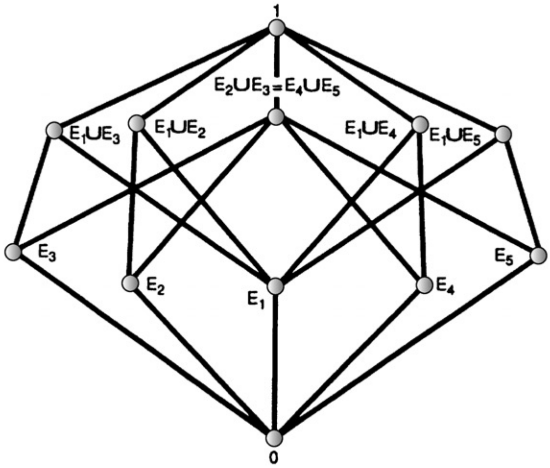

Fig. 1.

The smallest nontrivial case of this kind is depicted in Fig. 1. The graph represents two Boolean algebras $\mathscr{B}_1 = \langle E_1, E_2, E_3 \rangle$, $\mathscr{B}_2 = \langle E_1, E_4, E_5 \rangle$ corresponding to the outcomes of two incompatible measurements, and they share a common event E_1. The complement of E_1 denoted by $\bar{E}_1$ is identified as $E_2 \bigcup E_3 = E_4 \bigcup E_5$. The edges in the graph represent the partial order relations in each algebra from bottom to top. A realization of these relations can be obtained by the system considered in Kochen and Specker (1967): Let $S_x^2, S_{x'}^2, S_y^2, S_{y'}^2, S_z^2$ be the squared components of spin in the x, x', y, y', z directions of a spin-1 (massive) particle, where x, y, z and x', y', z form two orthogonal triples of directions with the z-direction in common. The operators S_x^2, S_y^2 and S_z^2 all commute, and have eigenvalues 0, 1. They can be measured simultaneously, and they satisfy $S_x^2 + S_y^2 + S_z^2 = 2I$. Similar relations hold in the other triple x', y', z. Now define $A = S_x^2 - S_y^2$, $B = S_z^2$, and $C = S_{x'}^2 - S_{y'}^2$. Then $E_1 = \{A = 0\} = \{B = 0\} = \{C = 0\}$, $E_2 = \{A = 1\}$, $E_3 = \{A = -1\}$, $E_4 = \{C = 1\}$, $E_5 = \{C = -1\}$ satisfy the relations in Fig. 1.

In sum, we assume that when the set of measurements $\mathscr{M}$ is announced, the agent is fully aware of the number of outcomes in each measurement and of the relations between the Boolean algebras they generate. In the spin-1 case just considered, it means that the agent is aware of the graph structure in Fig. 1. We shall refer in short to this background knowledge as *the logic of the gamble*. We assume no further knowledge on the part of the agent, in particular, no knowledge of quantum mechanics. Our purpose is to calculate the constraints on the probabilities that a rational agent can place in such gambles.

1.2. Methodological interlude: identity of observables and operational definitions

At this stage, one might object that the identity of observables in quantum mechanics *depends on probability*. Consider the case of the operators A, B, C such that $[A, B] = 0, [B, C] = 0$, but $[A, C] \neq 0$, and the two incompatible measurements of A together with B, and of B together with C. We are assuming that the agent is aware of the fact that the events $\{B = b_i\}$ are the same in both measurements. However, the actual procedure of measuring B can be very different in the two cases, so how does such awareness comes about? Indeed, the identity criterion for (our kind of) observables is: two procedures constitute measurements of the same observable if for any given physical state (preparation), they yield identical probability distribution over the set of possible outcomes.[2] It seems, therefore, that foreknowledge of the probabilities is a necessary condition for defining the identities of observables. But now we face a similar problem, how would one know when two *states* are the same? Identical states can be prepared in ways that are physically quite distinct. Well, two preparations yield the same state if for any given measurement they result in the same distribution of outcomes. It is a vicious circle.

There is nothing special about this circularity, a typical characteristic of operational "definitions" (Putnam, 1965). In fact, one encounters a similar problem in traditional probability theory in the interplay between the identity of events and their probability. The way to proceed is to remember that the point of the operational exercise is not to reduce the theoretical objects of the theory to experiments, but to *analyze* their meaning and their respective role in the theory. In this idealized and nonreductive approach, one takes the identity of one family of objects as somehow given and proceeds to recover the rest.

Consider how this is done in a recent article by Hardy (2001). Assuming that the probabilities of quantum measurements are experimentally given as relative frequencies, and assuming they satisfy certain relations, Hardy derives the structure of the observables (that is, the Hilbert space). His "solution" to the problem of the identity of states, or preparations, is simple. He stipulates that "preparation" corresponds to a position of a certain dial, one dial position for each preparation. The problem is simply avoided by idealizing it away.

Our approach is the mirror image of Hardy's. We are assuming that the identities of the observables (and in particular, events) are given, and proceed to recover the probabilities. This line of development is shared with all traditional approaches to probability where the identity of the events is invariably assumed to be given prior to the development of the theory. It is, moreover, easy to think of an idealized story which would cover our identity assumption. For example, in the three operator case A, B, C we can imagine that the results of their measurements are presented on three different dials. If B is measured together with A then the A-dial and B-dial show the results; if B and C are measured together the B-dial and C-dial show the results. Thus, fraud notwithstanding, the agent knows that he faces the measurement of the same B simply because the same gadget shows the outcome in both cases.

[2] In a deterministic world we would have a different criterion: two procedures constitute measurements of the same observable if for any given physical state they yield identical outcomes. We shall come back to this criterion in Section 3.2.

1.3. Rules of gambling

Our purpose is to calculate the constraints on the probabilities that a rational agent can place in a quantum gamble $\mathscr{M}$. These probabilities have the form $p(F|\mathscr{B})$ where $\mathscr{B} \in \mathscr{M}$ and $F \in \mathscr{B}$. The elements $F \in \bigcup_{\mathscr{B} \in \mathscr{M}} \mathscr{B}$ will be called simply "events." It is understood that an event is always given in the context of a measurement $\mathscr{B} \in \mathscr{M}$. The probability $p(F|\mathscr{B})$ is the degree of belief that the event F occurs in the measurement $\mathscr{B}$. There are two rules of rational gambling, the first is straightforward and the second more subtle.

RULE 1. *For each measurement $\mathscr{B} \in \mathscr{M}$ the function $p(\cdot|\mathscr{B})$ is a probability distribution on $\mathscr{B}$.*

This follows directly from the classical Bayesian approach. Recall that after the third stage in the quantum gamble the agent faces a bet on the outcome of a single measurement. The situation at this stage is essentially the same as tossing of a coin or casting of a die. Hence, the probability values assigned to the possible outcomes of the chosen measurement should be coherent. In other words, they have to satisfy the axioms of the probability calculus. The argument is that an agent who fails to be coherent will be compelled by the bookie to place bets that will cause him a sure loss (this is the "Dutch Book" argument). The argument is developed in detail in many texts (for example, de Finetti, 1972), and I will not repeat it here. Since at the outset the agent does not know which measurement $\mathscr{B} \in \mathscr{M}$ will be chosen by the bookie, RULE 1 follows.

RULE 2. *If $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{M}, F \in \mathscr{B}_1 \bigcap \mathscr{B}_2$ then $p(F|\mathscr{B}_1) = p(F|\mathscr{B}_2)$.*

The rule asserts the noncontextuality of probability (Barnum et al., 2000). It is not so much a rule of rationality, rather it is related to the logic of the gamble and the identity of observables (remembering that we identify each observable with the Boolean algebra generated by its possible outcomes).

Suppose that in the game $\mathscr{M}$, there are two measurements $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{M}$, and an event $F \in \mathscr{B}_1 \bigcap \mathscr{B}_2$. Assume that an agent chooses to assign $p(F|\mathscr{B}_1) \neq p(F|\mathscr{B}_2)$. A natural question to ask her, then, is why she assigns F different probabilities in the two contexts, though she thinks it is the same event. The only answer consistent with Bayesian probability theory is that she takes the $p(F|\mathscr{B}_i)$ as *conditional probabilities* and therefore not necessarily equal. In other words, she considers the act of choosing an experiment $\mathscr{B}_i$ (in stage 3 of the gamble) as an event in a larger algebra $\mathscr{B}$ which contains $\mathscr{B}_1$, $\mathscr{B}_2$. Consequently, she calculates the conditional probability of F, given the choice of $\mathscr{B}_i$.

There are two problems with this view. First, the agent can no longer maintain that $F \in \mathscr{B}_1 \bigcap \mathscr{B}_2$; in fact, F is not an element of any of the $\mathscr{B}_i$'s and can no longer be described as *an outcome* of a measurement. Second, the agent assumes that there is a single "big" Boolean algebra $\mathscr{B}$, the event F is an element of $\mathscr{B}$, and $\mathscr{B}_1$, $\mathscr{B}_2$ are subalgebras of $\mathscr{B}$. The trouble is that for sufficiently rich games $\mathscr{M}$, this assumption is inconsistent. In other words, there are gambles $\mathscr{M}$ which cannot be imbedded in a Boolean algebra without destroying the identities of the events and the logical relations between them. This is a consequence of the Kochen and Specker (1967) theorem to which we shall come in Section 2.2. It means, essentially, that an agent who violates RULE 2 is failing to grasp the logic of the gamble and wrongly assumes that she is playing a different game.

Another possibility is that assigning $p(F|\mathscr{B}_1) \neq p(F|\mathscr{B}_2)$ indicates that the agent is using a different notion of conditional probability. The burden of clarification is then on the agent to uncover her sense of conditionalization and show how it is related to quantum gambles. Thus, we conclude that the violation of RULE 2 implies either an ignorance of the logic of the gamble or an incoherent use of conditional probabilities. It is clear that our argument here is weaker than the Dutch book argument for RULE 1. A violation of RULE 2 does not imply a sure loss in a single shot game. We shall return to this argument, with a greater detail, in Section 2.2.

Rational probability values assigned in finite games need not be numerically identical to the quantum mechanical probabilities. However, with sufficiently complex gambles we can show that all the interesting features of quantum probability—from the uncertainty principle to the violation of Bell inequality—are present even in finite gambles. If we extend our discussion to gambles with an infinity of possible measurements, then RULEs 1 and 2 force the probabilities to follow the Born rule (Section 2.4).

1.4. Note on possible games

A quantum gamble is a set of Boolean algebras with certain (possible) relations between them. The details of these algebras and their relations is all that the agent needs to know. We do not assume that the agent knows any quantum theory.

However, engineers who construct gambling devices should know a little more. They should be aware of the physical possibilities. This is true in the classical domain as much as in the quantum domain. After all, the theory of probability, even in its most subjective form, associates a person's degree of belief with the objective possibilities in the physical world. In the quantum case the objective physical part concerns the *type* of gambles which can actually be constructed. It turns out that not all finite families of Boolean algebras represent possible games, at least as far as present day physics is concerned. I shall describe the family of possible gambles in a somewhat abstract way. It is a consequence of von Neumann (1955) analysis of the set of possible measurements.

Let $\mathbb{H}$ be the n-dimensional vector space over the real or complex field, equipped with the usual inner product. Let $H_1, H_2, \ldots, H_k$ be k nonzero subspaces of $\mathbb{H}$, which are orthogonal in pairs $H_i \perp H_j$ for $i,j = 1,2,\ldots,k$, and which together span the entire space, $H_1 \oplus H_2 \oplus \cdots \oplus H_k = \mathbb{H}$. These subspaces generate a Boolean algebra, call it $\mathscr{B}(H_1, H_2, \ldots, H_k)$, in the following way: The zero of the algebra is the null subspace, the nonzero elements of the algebra are subspaces of the form $H_{i_1} \oplus H_{i_2} \oplus \cdots \oplus H_{i_r}$ where $\phi \neq \{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, k\}$. If H, H' are two elements in the algebra, let $H \bigvee H' = H \oplus H'$ be the subspace spanned by the (set theoretic) union $H \bigcup H'$, let $H \bigwedge H' = H \bigcap H'$, and let the complement of H, denoted by $H^{\perp}$, be the subspace orthogonal to H such that $H \oplus H^{\perp} = \mathbb{H}$. Then $\mathscr{B}(H_1, H_2, \ldots, H_k)$ with the operations $\bigvee$, $\bigwedge$, $\perp$ is a Boolean algebra with 2^k elements. Note that a maximal algebra of this kind is obtained when we take all the H_i's to be one-dimensional subspaces (rays). Then $k = n$ and the algebra has 2^n elements.

Now, let $\mathbb{B}(\mathbb{H})$ be the family of *all* the Boolean algebras obtained from subspaces of $\mathbb{H}$ in the way described above. Obviously, If $\mathscr{B}_1, \mathscr{B}_2 \in \mathbb{B}(\mathbb{H})$ then $\mathscr{B}_1 \bigcap \mathscr{B}_2$ is also a Boolean algebra in $\mathbb{B}(\mathbb{H})$. We shall say that two subspaces G, H of $\mathbb{H}$ are *compatible in* $\mathbb{H}$ if there is $\mathscr{B} \in \mathbb{B}(\mathbb{H})$ such that $G, H \in \mathscr{B}$, otherwise G and H are incompatible. Two algebras $\mathscr{B}_1$, $\mathscr{B}_2$ are *incompatible in* $\mathbb{H}$ if there are subspaces $G \in \mathscr{B}_1$ and $H \in \mathscr{B}_2$ which are incompatible.

Possibility Criterion: *$\mathscr{M}$ is a possible quantum gamble if there is a finite dimensional complex or real Hilbert space $\mathbb{H}$ such that $\mathscr{M}$ is a finite family of Boolean algebras in $\mathbb{B}(\mathbb{H})$ which are incompatible in pairs.*

One could proceed with the probabilistic account disregarding this criterion and, in fact, go beyond what is known to be physically possible (see Svozil, 1998). We shall not do that, however, and all the games considered in this paper are physically possible. With each of the gambles to be discussed in this paper, we proceed in two stages. First, we present the Boolean algebras, their relations and the consequences for probability. Second, we prove that the gamble obeys the possibility criterion.

2. Consequences

2.1. Uncertainty relations

Consider the following quantum gamble $\mathscr{M}$ consisting of seven incompatible measurements (Boolean algebras), each generated by its three possible outcomes: $\langle E_1, E_2, F_2 \rangle$, $\langle E_1, E_3, F_3 \rangle$, $\langle E_2, E_4, E_6 \rangle$, $\langle E_3, E_5, E_7 \rangle$, $\langle E_6, E_7, F \rangle$, $\langle E_4, E_8, F_4 \rangle$, $\langle E_5, E_8, F_5 \rangle$. Note that some of the outcomes are shared by two measurements; these are denoted by the letter E. The other outcomes each belong to a single algebra and are denoted by F. As before, when two algebras share an event, they also share its complement so that, for example, $\bar{E}_1 = E_2 \cup F_2 = E_3 \cup F_3$, and similarly in the other cases. The logical relations among the generators are depicted in Fig. 2. This is *the compatibility graph* of the generators. Each node in the graph represents an outcome; two nodes are connected by an edge, if and only if the corresponding outcomes belong to a common algebra; each triangle represents the generators of one of the algebras.

We assume that the agent is aware of the seven algebras and the connections between them. By RULE 2 the probability he assigns to each event is independent of the Boolean algebra (measurement) which is considered, for example, $p(E_2 | \langle E_1, E_2, F_2 \rangle) = p(E_2 | \langle E_2, E_4, E_6 \rangle) \equiv p(E_2)$. RULE 1 entails that the probabilities of each triple of outcomes of each measurement should sum up to 1, for example, $p(E_4) + p(E_8) + p(F_4) = 1$. There are altogether seven equations of this

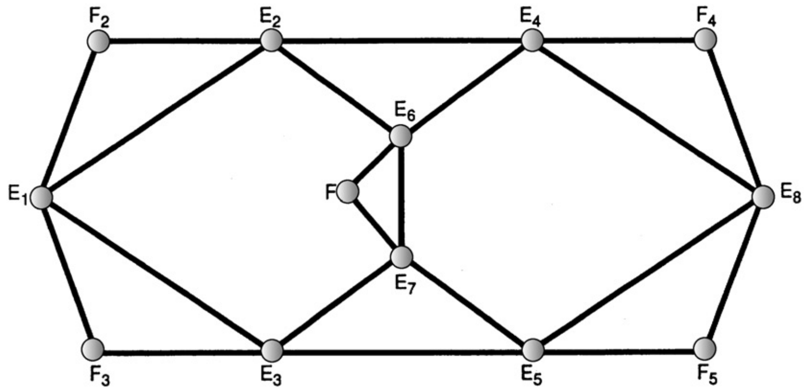

Fig. 2.

kind. Combining them with the fact that probability is nonnegative (by RULE 1), it is easy to prove that the probabilities assigned by our rational agent should satisfy $p(E_1) + p(E_8) \leqslant \frac{3}{2}$. This is an example of an *uncertainty relation*, a constraint on the probabilities assigned to the outcomes of incompatible measurements. In particular, if the system is prepared in such a way that it is rational to assign $p(E_1) = 1$ (see Section 2.5), then the rules of quantum games force $p(E_8) \leqslant \frac{1}{2}$.

To see why $\mathscr{M}$ represents a physically possible gamble, we use the *possibility criterion* and identify each event with a one-dimensional subspace of $\mathbb{C}^3$ (or $\mathbb{R}^3$) in the following way: E_1 is the subspace spanned by the vector (1, 0, 2), $E_2 \sim (0, 1, 0)$, $F_2 \sim (2, 0, -1)$, $E_3 \sim (2, 1, -1)$, $F_3 \sim (2, -5, -1)$, $E_4 \sim (0, 0, 1)$, $E_5 \sim (1, -1, 1)$, $E_6 = (1, 0, 0)$, $E_7 \sim (0, 1, 1)$, $F \sim (0, 1, -1)$, $F_4 \sim (1, -1, 0)$, $F_5 \sim (-1, 1, 2)$, $E_8 \sim (1, 1, 0)$. Note that the vectors associated with compatible subspaces are orthogonal, so that Fig. 2 is the orthogonality graph for these 13 vectors.

A more concrete way to represent this game is to consider each of these vectors as depicting a direction in physical space. For the vector v let S_v^2 be the square of the spin in the v-direction of a massive spin-1 particle, so that its eigenvalues are 0, 1. Now, for each of the 13 vectors above, take the event $\{S_v^2 = 0\}$. Then the relations in Fig. 2 are satisfied.

This example is a special case of a more general principle (Pitowsky, 1998):

Theorem 1. *Let H_1, H_2 be two incompatible rays in a Hilbert space $\mathbb{H}$ whose finite dimension is $\geqslant 3$. Then there is a (finite) quantum gamble $\mathscr{M} \subset \mathbb{B}(\mathbb{H})$ in which H_1, H_2 are events, and every probability assignment p for $\mathscr{M}$ which satisfies RULEs 1 and 2 also satisfies $p(H_1) + p(H_2) < 2$.*

2.2. Truth and probability; the Kochen and Specker's theorem

Consider the gamble $\mathscr{M}$ of eleven incompatible measurements, each with four possible outcomes:

$$\mathscr{B}_1 = \langle E_1, F_1, F_2, F_3 \rangle, \quad \mathscr{B}_2 = \langle E_1, F_1, F_4, F_5 \rangle, \quad \mathscr{B}_3 = \langle E_1, F_2, F_6, F_7 \rangle,$$
$$\mathscr{B}_4 = \langle E_1, F_3, F_8, F_9 \rangle, \quad \mathscr{B}_5 = \langle E_2, F_{10}, F_{11}, F_{12} \rangle, \quad \mathscr{B}_6 = \langle E_2, F_7, F_{10}, F_{13} \rangle,$$
$$\mathscr{B}_7 = \langle E_2, F_8, F_{11}, F_{14} \rangle, \quad \mathscr{B}_8 = \langle E_2, F_4, F_{12}, F_{15} \rangle, \quad \mathscr{B}_9 = \langle F_9, F_{14}, F_{16}, F_{17} \rangle,$$
$$\mathscr{B}_{10} = \langle F_5, F_{15}, F_{16}, F_{18} \rangle, \quad \mathscr{B}_{11} = \langle F_6, F_{13}, F_{17}, F_{18} \rangle.$$

The two outcomes denoted by the letter E are shared by four measurements each, and the outcomes denoted by F are shared by two measurements each. Altogether, there are twenty outcomes. This example is based on a proof of the Kochen and Specker (1967) theorem due to Kergnahan (1994). (The original proof requires hundreds of measurements, with three outcomes each.) Again, when an event is shared by two measurements then so is its complement, for example, $\bar{F}_8 = E_1 \cup F_3 \cup F_9 = E_2 \cup F_{11} \cup F_{14}$.

Now, suppose that all the algebras $\mathscr{B}_k$ are subalgebras of a single Boolean algebra $\mathscr{B}$. Assume, without loss of generality, that $\mathscr{B}$ is an algebra of subsets of a set X. With this identification, the events E_i, F_j are subsets of X. The logical relations between the events dictates that any two of the events among the E_i's and F_j's that share the same algebra $\mathscr{B}_k$ are disjoint. Moreover, the union of all four outcomes in each algebra $\mathscr{B}_k$ is identical to X; for example, $X = E_2 \bigcup F_7 \bigcup F_{10} \bigcup F_{13}$ is the union of the outcomes in $\mathscr{B}_6$. But this leads to a contradiction because the intersection of all these unions is necessarily empty!

To see that, suppose, by contrast, that there is $x \in X$ such that x belongs to exactly one outcome, E_i or F_j, in each one of the 11 algebras $\mathscr{B}_k$. This means that x belongs to eleven such events (with repetition counted). But this is impossible since each one of the outcomes appears an even number of times in the 11 algebras, and 11 is an odd number.

One consequence of this is related to RULE 2, discussed in Section 1.3. Suppose that an agent thinks about the probabilities of the events E_i, F_j as conditional on the measurement performed. If the term "conditional probability" is used in its usual sense, then the events should be interpreted as elements of a single Boolean algebra $\mathscr{B}$ (taken again as an algebra of subsets of some set X). To avoid the Kochen–Specker contradiction, the agent can use two strategies. The first is to take some of the generating events in at least one algebra to be nondisjoint in pairs; for example, $E_2 \bigcap F_8 \neq \phi$. In this case the agent ceases to see the events E_2, F_8 as representing *measurement outcomes* and associates with them some other meaning (although he eventually takes the conditional probability of $E_2 \bigcap F_8$ to be zero). The other strategy is to take the union of the outcomes of some measurements $\mathscr{B}_k$ to be proper subsets of X. For example, in the case of $\mathscr{B}_9$, the agent assumes $F_9 \bigcup F_{14} \bigcup F_{16} \bigcup F_{17} \subsetneq X$. In this case, he actually adds another theoretical outcome to $\mathscr{B}_9$ (which, however, has conditional probability zero). Both strategies represent a distortion of the logical relations among the events which we have assumed as given.

On a less formal, level, we can ask, "why would anyone do that?" The additional structure assumed by the agent amounts to a strange "hidden variable theory" for the set of experiments $\mathscr{M}$. There is a great theoretical interest in hidden variable theories, but they are of little value to the rational gambler. A classical analogue would be a person who thinks that a coin *really* has three sides "head," "belly," and "tail" and assigns a prior probability $\frac{1}{3}$ to each. But the act of tossing the coin (or looking at it, or physically interacting with it) causes the belly side never to show up, so the probability of belly, conditional on tossing (or looking, or interacting), is zero. The betting behavior of such a person is rational in the sense that no Dutch book argument against him is possible. However, as far as gambling on a coin toss is concerned, his theory of coins is not altogether rational. It is the elimination of this kind of irrationality which motivates RULE 2.

Another consequence of this gamble concerns the relations between probability and logical truth. Often the Kochen and Specker theorem is taken as an indication that in quantum mechanics a classical logical falsity may sometimes be true (Bub, 1974; Demopoulos, 1976). To see how, consider the E_i and F_j as *propositional variables*, and for each $1 \leqslant k \leqslant 11$ let C_k be the proposition which says: "exactly one of the variable in the group k is true;" for example,

$$C_6 = (E_2 \vee F_7 \vee F_{10} \vee F_{13}) \wedge \sim(E_2 \wedge F_7) \wedge \sim(E_2 \wedge F_{10}) \wedge \sim(E_2 \wedge F_{13}) \wedge \sim(F_7 \wedge F_{10}) \wedge \sim(F_7 \wedge F_{13}) \wedge \sim(F_{10} \wedge F_{13}).$$

Then $\bigwedge_{k=1}^{11} C_k$ is a classical logical falsity. But $\bigwedge_{k=1}^{11} C_k$ is "quantum mechanically true" with respect to the system described above, because each one of the C_k's is a true description of it.

In our gambling picture, we make a more modest claim. A rational agent who participates in the quantum gamble will assign, in advance, probability 1 to each C_k. Therefore, arguably the agent also assigns $\bigwedge_{k=1}^{11} C_k$ probability 1. But this is an *epistemic* position which does not oblige the agent to assign truth values to the E_i's and F_j's, nor is he committed to say that such truth values exist. Indeed, this is a strong indication that "probability one" and "truth" are quite different from one another. The EPR system (below) provides another example. There is, however, a weaker sense in which $\bigwedge_{k=1}^{11} C_k$ is true and we shall discuss it in the philosophical discussion (Section 3.1).

The following is a proof that our game satisfies the *possibility criterion*. Each E_i and each F_j is identified with a ray (one-dimensional subspace) of $\mathbb{C}^4$ (or $\mathbb{R}^4$). Two outcomes which share the same algebra correspond to orthogonal rays. The rays are identified by a vector that spans them:

$$E_1 \sim (1,0,0,0),\quad F_1 \sim (0,1,0,0),\quad F_2 \sim (0,0,1,0),\quad F_3 \sim (0,0,0,1),$$
$$F_4 \sim (0,0,1,1),\quad F_5 \sim (0,0,1,-1),\quad F_6 \sim (0,1,0,1),\quad F_7 \sim (0,1,0,-1),$$
$$F_8 \sim (0,1,1,0),\quad F_9 \sim (0,1,-1,0),\quad E_2 \sim (1,1,-1,1),\quad F_{10} \sim (-1,1,1,1),$$
$$F_{11} \sim (1,-1,1,1),\quad F_{12} \sim (1,1,1,-1),\quad F_{13} \sim (1,0,1,0),\quad F_{14} \sim (1,0,0,-1),$$
$$F_{15} \sim (1,-1,0,0),\quad F_{16} \sim (1,1,1,1),\quad F_{17} \sim (1,-1,-1,1),\quad F_{18} \sim (1,1,-1,-1).$$

2.3. EPR and violation of Bell's inequality

Given two (not necessarily disjoint) events A, B in the same algebra, denote $AB = A \bigcap B$, and for three events A, B, C denote by $\{A, B, C\}$ the Boolean algebra that they generate:

$$\{A, B, C\} = \langle ABC,\ \bar{A}BC,\ A\bar{B}C,\ AB\bar{C},\ \bar{A}\bar{B}C,\ \bar{A}B\bar{C},\ A\bar{B}\bar{C},\ \bar{A}\bar{B}\bar{C} \rangle.$$

In order to recover the argument of the Einstein, Rosen, and Podolsky (1935) and Bell (1964) paradox within a quantum gamble, we shall use Mermin's (1990) representation of GHZ, the Greenberger, Horne, and Zeilinger (1989) system. Consider the gamble which consists of eight possible measurements: The four measurements $\mathscr{B}_1 = \{A_1, B_1, C_1\}$, $\mathscr{B}_2 = \{A_1, B_2, C_2\}$, $\mathscr{B}_3 = \{A_2, B_1, C_2\}$, $\mathscr{B}_4 = \{A_2, B_2, C_1\}$, each with eight possible outcomes, and

$$\mathscr{B}_5 = \langle S,\ D_1,\ A_1B_1C_1,\ \bar{A}_1\bar{B}_1C_1,\ \bar{A}_1B_1\bar{C}_1,\ A_1\bar{B}_1\bar{C}_1 \rangle,$$
$$\mathscr{B}_6 = \langle S,\ D_2,\ \bar{A}_1B_2C_2,\ A_1\bar{B}_2C_2,\ A_1B_2\bar{C}_2,\ \bar{A}_2\bar{B}_1\bar{C}_2 \rangle,$$
$$\mathscr{B}_7 = \langle S,\ D_3,\ \bar{A}_2B_1C_2,\ A_2\bar{B}_1C_2,\ A_2B_1\bar{C}_2,\ \bar{A}_2\bar{B}_1\bar{C}_2 \rangle,$$
$$\mathscr{B}_8 = \langle S,\ D_4,\ \bar{A}_2B_2C_1,\ A_2\bar{B}_2C_1,\ A_2B_2\bar{C}_1,\ \bar{A}_2\bar{B}_2\bar{C}_1 \rangle,$$

each with six possible outcomes.

Assume that the agent has good reasons to believe that $p(S) = 1$. Such a belief can come about in a variety of ways; for example, she may know something about the preparation of the system from a previous measurement result (see Section 2.5). Alternatively, the bookie may announce in advance that he will raise his stakes indefinitely against any bet made for $\bar{S}$. Whatever the source of information, the agent has good reasons to assign probability zero to four out of the eight outcomes in each one of the four measurements $\mathscr{B}_1$ to $\mathscr{B}_4$. The remaining events are

$$\begin{aligned}
&\text{in } \mathscr{B}_1\ \ \bar{A}_1B_1C_1,\ A_1\bar{B}_1C_1,\ A_1B_1\bar{C}_1,\ \bar{A}_1\bar{B}_1\bar{C}_1\\
&\text{in } \mathscr{B}_2\ \ A_1B_2C_2,\ \bar{A}_1\bar{B}_2C_2,\ \bar{A}_1B_2\bar{C}_2,\ A_1\bar{B}_2\bar{C}_2\\
&\text{in } \mathscr{B}_3\ \ A_2B_1C_2,\ \bar{A}_2\bar{B}_1C_2,\ \bar{A}_2B_1\bar{C}_2,\ A_2\bar{B}_1\bar{C}_2\\
&\text{in } \mathscr{B}_4\ \ A_2B_2C_1,\ \bar{A}_2\bar{B}_2C_1,\ \bar{A}_2B_2\bar{C}_1,\ A_2\bar{B}_2\bar{C}_1
\end{aligned} \tag{1}$$

Denote by P the sum of the probabilities of these 16 events. Given that $p(S) = 1$, the probabilities of the events in each row in (1) sum up to 1. Altogether, the rational assignment is therefore $P = 4$. However, if A_1, B_1, C_1, A_2, B_2, C_2 are events in any (classical) probability space then the sum of the probabilities of the events in (1) never exceeds 3. This is one of the constraints on the values of probabilities which Boole called "conditions of possible experience,"[3] and it is violated by any rational assignment in this quantum gamble. On one level this is just another example of a quantum gamble that cannot be imbedded in a single classical probability space without distorting the identity of the events and the logical relations between them. A more dramatic example has been the Kochen and Specker's theorem of the previous section.

The special importance of the EPR case lies in the details of the physical system and the way the measurements $\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_4$ are performed. The system is composed of three particles which interacted in the past but are now spatially

[3] See Pitowsky (1989, 1994, 2002) and Pitowsky and Svozil (2001) for a discussion of Boole's conditions, their derivations and their violations by quantum frequencies.

separated and are no longer interacting. On the first particle, we can choose to perform an A_1-measurement or an A_2-measurement (but not both), each with two possible outcomes. Similarly, we can choose to perform on the second particle one of two B-measurements, and one of two C-measurements on the third particle. The algebras $\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3$, $\mathscr{B}_4$ represent the outcomes of four out of the eight logically possible combinations of such local measurements. In this physical arrangement, we can recover the EPR reasoning, and Bell's rebuttal, which I will not repeat here. The essence of Bell's theorem is that the EPR assumptions lead to the conclusion that $A_1, B_1, C_1, A_2, B_2, C_2$ belong to a single Boolean algebra. Consequently, the sum of the probabilities of the events in (1) cannot exceed 3, in contradiction to RULEs 1 and 2.

Which of two EPR assumptions "reality" or "locality" should the Bayesian reject? In the previous section, we have made the distinction between "probability 1" and "truth." But the *identification* of the two is precisely the subject matter of EPR's Principle of Reality: "If without in any way disturbing a system we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to this physical quantity" (Einstein et al., 1935). Quite independently of Bell's argument, a Bayesian should take a sceptical view of this principle. "Probability equal to unity" means that the degree of rational belief has reached a level of certainty. It does not reflect any prejudice about possible causes of the outcomes. On the other hand, there seem to be no good grounds for rejecting the Principle of Locality on the basis of this or similar gambles.

To prove that this gamble satisfies the possibility criterion, let $\mathbb{H}_2$ be the two-dimensional complex Hilbert space, let σ_x, σ_y be the Pauli matrices associated with the two orthogonal directions x, y, and let H_x, H_y the (one-dimensional) subspaces of $\mathbb{H}_2$ corresponding to the eigenvalues $\sigma_x = 1$, $\sigma_y = 1$, respectively, so that $H_x^{\perp}$, $H_y^{\perp}$ correspond to $\sigma_x = -1$, $\sigma_y = -1$. In the eight-dimensional Hilbert space $\mathbb{H}_2 \otimes \mathbb{H}_2 \otimes \mathbb{H}_2$, we shall identify $A_1 = H_x \otimes \mathbb{H}_2 \otimes \mathbb{H}_2$, $B_1 = \mathbb{H}_2 \otimes H_x \otimes \mathbb{H}_2$, $C_1 = \mathbb{H}_2 \otimes \mathbb{H}_2 \otimes H_x$, $A_2 = H_y \otimes \mathbb{H}_2 \otimes \mathbb{H}_2$, $B_2 = \mathbb{H}_2 \otimes H_y \otimes \mathbb{H}_2$, $C_2 = \mathbb{H}_2 \otimes \mathbb{H}_2 \otimes H_y$; all these are four-dimensional subspaces. The outcomes in $\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_4$ are one-dimensional subspaces; for example, $\bar{A}_1 \bar{B}_2 C_2 = H_x^{\perp} \otimes H_y^{\perp} \otimes H_y$. The subspace S is the one-dimensional ray along the GHZ state $\sqrt{1/2}(|+_z\rangle_1 |+_z\rangle_2 |+_z\rangle_3 - |-_z\rangle_1 |-_z\rangle_2 |-_z\rangle_3)$ where z is the direction orthogonal to x and y. The subspaces D_i are just the orthocomplements, in $\mathbb{H}_2 \otimes \mathbb{H}_2 \otimes \mathbb{H}_2$, to the direct sum of the other subspaces in their respective algebras. Hence, $\dim D_i = 3$.

2.4. The infinite gamble: Gleason's theorem

Let us take the idealization a step further. Assume that the bookie announces that $\mathscr{M}$ contains all the maximal Boolean algebras in $\mathbb{B}(\mathbb{H})$ for some finite dimensional real or complex Hilbert space $\mathbb{H}$ with $\dim \mathbb{H} = n \geqslant 3$. Recall that if $H_1, H_2, \ldots, H_k$ are k nonzero subspaces of $\mathbb{H}$, which are orthogonal in pairs, and whose direct sum is the entire space, they generate a Boolean algebra $\mathscr{B}(H_1, H_2, \ldots, H_k)$ (Section 1.4). If $k = n$ the algebra is maximal, and each subspace H_j is one-dimensional. In other words, the set $\mathscr{M}$ consists of all nondegenerate measurements with n outcomes. The information theoretic aspects of this case are discussed in Caves et al. (2002).

There is a certain difficulty in extending quantum gambles to this case since there are a continuum of possible measurements, and the agent is supposed to place money on each. We can overcome this difficulty by assuming that the agent makes a commitment to pay a certain amount on each outcome of each measurement without paying any cash in advance. When a single measurement $\mathscr{B} \in \mathscr{M}$ is chosen by the bookie, all the agent's commitments are canceled except those pertaining to $\mathscr{B}$.

RULEs 1 and 2 imply in this case that for any n orthogonal rays $H_1, H_2, \ldots, H_n$ in $\mathbb{H}$, the agent's probability function should satisfy

$$p(H_1) + p(H_2) + \cdots + p(H_n) = 1. \tag{2}$$

Gleason (1957) proved

Theorem 2. *Let $\mathbb{H}$ be a Hilbert space over field of real or complex numbers with a finite dimension $n \geqslant 3$. If p is a nonnegative function defined on the subspaces of $\mathbb{H}$ and satisfies* (2) *for every set of n orthogonal rays then there is a statistical operator W such that for every subspace H of $\mathbb{H}$*

$$p(H) = tr(WP_H), \tag{3}$$

where P_H is the projection operator on H.

(For the proof, see also Pitowsky, 1998.) This profound theorem gives a characterization of all probability assignments of quantum theory. Furthermore, if we know that the system is prepared with $p(R) = 1$, for some ray R, then p is uniquely determined by $p(H) = \|P_H(r)\|^2$ for all subspaces H, where r is a unit vector that spans R. The theorem can be easily extended to closed subspaces of the infinite dimensional Hilbert space.

Many of the results about finite quantum gambles that we have considered are actually consequences of Gleason's theorem. Consider, for example, the Kochen and Specker's theorem (Section 2.2). To connect it with Gleason's theorem, take an appropriate first-order formal theory of the rays of $\mathbb{R}^n$, the orthogonality relation between them, and the real functions defined on them (where $n \geqslant 3$ finite and fixed). Add to it a special function symbol p, the axiom that p is nonnegative, the axiom that p is not a constant, the axiom that p has only two values zero or one. Now, add the infinitely many axioms $p(H_1) + p(H_2) + \cdots + p(H_n) = 1$ for each n-tuple of orthogonal rays in $\mathbb{R}^n$. By Gleason's theorem, this theory is inconsistent (since by (3) p has a continuum of values). Hence, there is a finite subset of this set of axioms which is inconsistent, meaning a finite subset of rays which satisfy the Kochen and Specker's theorem. This is, of course, a nonconstructive proof, and an explicit construction is preferable. However, the consideration just mentioned can be used to obtain more general nonconstructive results about finite games. One such immediate result is Theorem 1 which also has a constructive proof. In fact, the proof of Gleason's theorem involves a construction similar to that of Theorem 1 (see Pitowsky, 1998).

Gleason's theorem indicates that the use of the adjective "subjective" to describe epistemic probability is a misnomer. Even in the classical realm it has misleading connotations. Classically, different agents that start with different prior probability assignments eventually converge on the same probability distribution as they learn more and more from common experience. In the quantum realm the situation is more extreme. For a given single physical system, Gleason's theorem dictates that all agents share a common prior probability distribution or, in the worst case, they start using the same probability distribution after a single (maximal) measurement.

2.5. A note on conditional quantum probability

Consider two gambles, $\mathcal{M}_1$, $\mathcal{M}_2$, and assume that A is a common event. In other words, there is $\mathcal{B}_1 \in \mathcal{M}_1$ and $\mathcal{B}_2 \in \mathcal{M}_2$ such that $A \in \mathcal{B}_1 \bigcap \mathcal{B}_2$. We can consider sequential gambles in which the gamble $\mathcal{M}_1$ is played, and after the results are recorded, the gamble $\mathcal{M}_2$ follows with the measurements performed *on the same system*. In such cases, the agent can place *conditional bets* of the form: "If A occurs in the first gamble place such and such odds in the second gamble." This means that the probabilities assigned in the second game $\mathcal{M}_2$ are constrained by the condition $p(A) = 1$ (in addition to the constraints imposed by RULEs 1 and 2). The EPR gamble in Section 2.3 can be seen as such a conditional game when we consider the preparation process as a previous gamble with an outcome S. In fact, all preparations (at least of pure states) can be seen in that light.

If the gambles $\mathcal{M}_1$, $\mathcal{M}_2$ are infinite and contain all the maximal algebras in $\mathbb{B}(\mathbb{H})$, Gleason's theorem dictates the rule for conditional betting. In the second gamble, the probability is proportional to the square of the length of the projection on (the subspace corresponding to) A. The conditional probability is, therefore, given by Lüders rule (Bub, 1997).

3. Philosophical remarks

3.1. Semantics for quantum logic and structural realism

The line we have taken has some affinity with Bohr's approach—or more precisely, with the view often attributed to Bohr[4]—in that we treat the outcomes of future measurements as mere possibilities and do not associate them with properties that exist prior to the act of measurement. Bohr's position, however, has some other features which are better avoided. Consider a spin-1 massive particle and suppose that we measure S_z, its spin along the z-direction. Bohr would say that in this circumstance, attributing values to S_x and S_y is meaningless. But the equation $S_x^2 + S_y^2 + S_z^2 = 2I$ remains valid then, as it is valid at all times. How can an expression

[4] See Beller (1999). Although Bohr kept changing his views and contradicted himself on occasion, it is useful to distill from his various pronouncements a more or less coherent set. This is what philosophers mean by "Bohr's view."

which contains meaningless (or valueless) terms be itself valid? Indeed, noncommuting observables may satisfy algebraic equations; the Laws of Nature often take such form. What is the status of such equations at the time when only one component in them has been meaningfully assigned a value? What is their status when no measurement has been performed? Quantum logic, in some of its formulations, has been an attempt to answer this question *realistically.*

It began with the seminal work of Birkhoff and von Neumann (1936). A later modification was inspired by the work of Kochen and Specker (1967). The realist interpretation of the quantum logical formalism is due to Finkelstein (1962), Putnam (1968), Bub (1974), and Demopoulos (1976). Consider, for example, the gamble $\mathscr{B}_1 = \langle E_1, E_2, E_3 \rangle$, $\mathscr{B}_2 = \langle E_1, E_4, E_5 \rangle$ made of two incompatible measurements, with one common outcome E_1 (Fig. 1). Let us loosely identify the outcomes E_i with the propositions that describe them. The realist quantum logician maintains that both $E_1 \vee E_2 \vee E_3$ and $E_1 \vee E_4 \vee E_5$ are true, and, therefore, so is $A = (E_1 \vee E_2 \vee E_3) \wedge (E_1 \vee E_4 \vee E_5)$. But only one of the measurements $\mathscr{B}_1$ or $\mathscr{B}_2$ can be conducted at one time. This means that, generally, only three out of the five E_i's can be experimentally assigned a truth value (except in the case that E_1 turns out to be true which makes the other four events false). This does not prevent us from assigning *hypothetical* truth values to the E_i's that make A true. However, as we have seen in Section 2.2, the trouble begins when we consider more complex gambles. To repeat, let $\mathscr{M}$ be the gamble of Section 2.2, and for each $1 \leqslant k \leqslant 11$ let C_k be the proposition which says: "exactly one of the variables in the group k is true"; for example,

$$C_6 = (E_2 \vee F_7 \vee F_{10} \vee F_{13}) \wedge \sim(E_2 \wedge F_7) \wedge \sim(E_2 \wedge F_{10}) \\ \wedge \sim(E_2 \wedge F_{13}) \wedge \sim(F_7 \wedge F_{10}) \wedge \sim(F_7 \wedge F_{13}) \wedge \sim(F_{10} \wedge F_{13}).$$

Then $B = \bigwedge_{k=1}^{11} C_k$ is a classical logical falsity. This means that we cannot make B true even by assigning hypothetical truth values to the E_i's and F_j's.

Still, the quantum logician maintains that B is true. Or, by analogy, that $S_x^2 + S_y^2 + S_z^2 = 2I$ is true for every orthogonal triple x, y, z in physical space. This is the quantum logical solution of the Bohrian dilemma, and it comes with a heavy price-tag: the repudiation of classical prepositional logic. But what does it mean to say that B is true? As I have shown elsewhere (Pitowsky, 1989), the operational analysis of the quantum logical connectives, due to Finkelstein and Putnam, leads to a nonlocal hidden variable theory in disguise. Moreover, from a Bayesian perspective it is quite sufficient to say that B has probability 1, meaning that each conjunct in B has probability 1; that is, a degree of belief approaching certainty. Indeed, the Bayesian does not consider even the Laws of Nature as true, only as being nearly certain, given present day knowledge.

Nevertheless, there is a sense in which A or even B are true, and this is the sense that enables our Bayesian analysis in the first place. Thus, to assert that "$(E_1 \vee E_2 \vee E_3) \wedge (E_1 \vee E_4 \vee E_5)$ *is true*" is nothing but a cumbersome way to say that the gamble $\mathscr{M} = \{\langle E_1, E_2, E_3 \rangle, \langle E_1, E_4, E_5 \rangle\}$ exists. This is first and foremost a statement about the identities: the outcome E_1 is *really* the same in the two measurements, and $\bar{E}_1 = E_2 \vee E_3 = E_4 \vee E_5$. It is also a statement about physical realizations, this gamble can be designed and played (experimental difficulties notwithstanding). Viewed in this light quantum gambles together with RULEs 1 and 2 form *semantics for quantum logic*, in that they assign meaning to the identities of quantum logic (in its partial Boolean algebra formulation).

The metaphysical assumption underlying the Bayesian approach is therefore *realism about the structure of quantum gambles*, in particular those that satisfy the possibility criterion (Section 1.4). This position is close in spirit (but not identical) to the view that quantum mechanics is a complete theory, so let us turn to the alternative view.

3.2. Hidden variables—A Bayesian perspective

Consider Bohm's theory as a typical example.[5] Recall that in this theory, the state of a single particle at time t is given by the pair $(x(t), \psi(x,t))$ where x is the position of the particle and $\psi = R\exp(iS)$—the guiding wave—is a solution of the time dependent Schrödinger's equation. The guiding condition $m\dot{x} = \nabla S$ provides the relation between the two components of the state, where m is the particle mass. The theory is deterministic; an initial position $x(0)$ and an initial condition $\psi(x,0)$ determine the trajectory of the particle and the guiding wave at all future times. In particular, the outcome of every measurement is determined by these initial conditions.

As can be expected from the Kochen and Specker's theorem, the outcome of a measurement is context dependent in Bohm's theory. This fact can also be derived by a direct calculation (Pagonis & Clifton, 1995). Given a fixed initial state $(x(0), \psi(x,0))$, the measurement of S_z^2 together with S_x^2 and S_y^2 can yield one result $S_z^2 = 0$; but the measurement of S_z^2 together with $S_{x'}^2$ and $S_{y'}^2$ can give another result $S_z^2 = 1$. Now, the identity criterion for observables in a deterministic theory is: Two procedures constitute measurements of the same observable if for any given physical state (preparation) they yield identical outcomes. Therefore, in Bohm's theory, the observable "S_z^2 in the x, y, z context" is not really the same as "S_z^2 in the x', y', z context." Nevertheless, the Bohmians consider S_z^2 as one single statistical observable across contexts because the *average* outcome of S_z^2 over different initial positions with density $|\psi(x,0)|^2$ is context independent. Hence, Bohm's theory is a hybrid much like classical statistical mechanics: the dynamics are deterministic but the observables are statistical averages. Since the initial positions are not known—not even knowable—the averages provide the empirical content. Consequently, the observable structure of quantum mechanics is accepted by the Bohmians "for all practical purposes."

This attitude prevails when the Bohmian is betting in a quantum gamble. There is no detectable difference in the betting behavior of a Bohmian agent, although the reasons leading to his behavior follow from the causal structure of Bohm's theory. At a first glance there seems to be nothing peculiar about this. Many people who

[5] The uniqueness theorem (Bub & Clifton, 1996; Bub, 1997; Bub, Clifton, & Goldstein, 2000) implies that all "no collapse" hidden variable theories have essentially the structure of Bohm's theory.

would assign probability 0.5 to "heads" believe that the tossing of a coin is a deterministic process. Indeed, there is a rational basis to this belief: if the agent is allowed to inspect the initial conditions of the toss with a greater precision, he may change his betting odds. In other words, his 0.5 degree of belief is conditional on his lack of knowledge of the initial state. Obtaining further information is possible, in principle, and in the limit of infinite precision, it leads to the assignment of probability zero or one to "heads." For the Bayesian, this is in a large measure what determinism *means.*

Can we say the same about the Bohmian attitude in a quantum gamble? According to Bohm's theory[6] the position of the particle cannot be known beyond the information invested in the distribution $|\psi|^2$. Suppose that a particle is prepared in a (pure) quantum state $\psi(x,0)$. Then, any attempt to obtain further information about the value of the initial position will cause a change in ψ, and, therefore, also in the initial position itself. Hence, there is no process which, in the limit, yields certain knowledge of the initial position, and the analogy with classical determinism breaks down. Consequently, from a Bayesian perspective, the determinism of Bohm's theory is a myth, although it does not lead its believers astray in their bets.

What is the function of this myth? Obviously, it is to retain a sense of determinism, albeit one which is completely disconnected from human knowledge. But there is also a subtler issue here that has to do with the structure of the observables. As we have noticed, for the Bohmian, the event $E_1 = \{S_z^2 = 0$ in the x, y, z context$\}$ is not the same as the event $E_1' = \{S_z^2 = 0$ in the x', y', z context$\}$. Hence, the gamble $\mathscr{M} = \{\langle(E_1, E_2, E_3\rangle, \langle E_1, E_4, E_5\rangle\}$, is interpreted by him as being "really" $\mathscr{M}' = \{\langle E_1, E_2, E_3\rangle, \langle E_1', E_4, E_5\rangle\}$ although, as a result of dynamical causes, the long term frequencies of E_1 and E_1' happen to be identical (for any given fixed ψ). It follows that the myth also serves the purpose of "saving classical logic" by dynamical means (Pitowsky, 1994). Nowhere is this more apparent than in the EPR case where Bohm's dynamics violate locality on the level of individual processes.

In this sense, the hidden variable approach is conservative. It is not so much its insistence on determinism but rather the refusal to acknowledge that the structure of the set of events—our quantum gambles—is real. As a gambler, the Bohmian bets as if it is very real; as a metaphysician, he provides a complicated apology.

3.3. *Instrumentalism and its radical foundations*

The Bayesian approach represents an instrumental attitude towards the quantum state. The state is just a code for probabilities, and "probability theory is simply the quantitative formulation of how to make rational decisions in the face of uncertainty" (Fuchs & Peres, 2000). Instrumentalism seems metaphysically innocent, all we are dealing with are experiments and their outcomes; without a commitment to an underlying, completely described microscopic reality. One might even be

[6] Vallentini (1996) considers the possibility that $|\psi|^2$ is only an "equilibrium" distribution, and deviations from it are possible. In this case, Bohm's theory is a genuine empirical extension of quantum mechanics, and the Bohmian agent may sometimes bet against the rules of quantum mechanics.

tempted to think that "quantum theory needs no interpretation" (Fuchs & Peres, 2000). Of course, there is a sense in which this is true. One needs no causal picture to do physics. Like a gambler, the physicist can assign probabilities to outcomes, assuming no causal or other mechanisms which bring them about.

But instrumentalism simply pushes the question of interpretation one step up the ladder. Instead of dealing directly with "reality," the instrumentalist faces the challenge of explaining his instrument; that is, quantum probability. Unlike other mathematical theories—group theory for example—the application of probability requires a philosophical analysis. After all, probability theory is our tool for weighing the relative merits of alternative actions and for making *rational* decisions; decisions that are made rational by their justifications. Indeed, we have provided a part of the justification by demonstrating how the structure of quantum gambles, together with the gambling rules, dictate certain constraints on the assignment of probability values. The trouble is that these probability values violate classical constraints; for example, Bell's inequalities. One hundred and fifty years ago, Boole considered these and other similar constraints as "conditions of possible experience," and consequently conditions of rational choice. Today, we witness the appearance of "impossible" experience. The Bohmian explains it away by reference to unobservable nonlocal measurement disturbances. The instrumentalist, in turn, insists that there is nothing to explain. But the violations of the classical constraints (unlike the measurement disturbances) are provably real. Therefore, something should be said about it if we insist that "probability theory is simply the quantitative formulation of how to make rational decisions."

Instrumentalists often take their "raw material" to be the set of space–time events: clicks in counters, traces in bubble chambers, dots on photographic plates, and so on. Quantum theory imposes on this set a definite structure. Certain blips in space–time are identified as instances of the same event. Some families of clicks in counters are assumed to have logical relations with other families, etc. What we call *reality* is not just the bare set of events, *it is this set together with its structure*, for all that is left without the structure is noise. It has been von Neumann's great achievement to identify this structure and derive some of the consequences that follow from its details. I believe that von Neumann's contribution to the foundations of quantum theory is exceedingly more important than that of Bohr. It is one thing to say that the only role of quantum theory is to "predict experimental outcome" and that different measurements are "complementary." It is quite another thing to provide an *understanding* of what it means for two experiments to be incompatible, and yet for their possible outcomes to be related; to show how these relations imply the uncertainty principle; and even, finally, to realize that the structure of events dictates the numerical values of the probabilities (Gleason's theorem).

Bohr's position will not suffice even for the instrumentalists. Their view, far from being metaphysically innocent, is founded on an assumption which is more radical than that of the hidden variable theories. Namely, the taxonomy of the universe expressed in the structure of the set of possible events, the quantum gambles which are made possible, and the theory of probability they imply, are new and only partially understood pieces of knowledge. It is the task of an interpretation of quantum mechanics to make sense of these structures and relate them to what we previously called "probability" and even "logic."[7]

Acknowledgements

I would like to thank Harvey Brown, Jeremy Butterfield, William Demopoulos, Wayne Myrvold, and Simon Saunders for helpful comments and suggestions. This research is supported by an Israel Science Foundation grant number 879/02.

References

Barnum, H., Caves, C. M., Finkelstein, J., Fuchs, C. A., & Schack, R. (2000). Quantum probability from decision theory? *Proceedings of the Royal Society of London A*, *456*, 1175–1182.

Bell, J. S. (1964). On the Einstein–Podolsky–Rosen Paradox. *Physics*, *1*, 195–200.

Beller, M. (1999). *Quantum dialogue*. Chicago: The University of Chicago Press.

Birkhoff, G., & von Neumann, J. (1936). The logic of quantum mechanics. *Annals of Mathematics*, *37*, 823–843.

Bub, J. (1974). *The interpretation of quantum mechanics*. Dordrecht: Reidel.

Bub, J. (1997). *Interpreting the quantum world*. Cambridge: Cambridge University Press.

Bub, J., & Clifton, R. (1996). A uniqueness theorem for "No Collapse" interpretations of quantum mechanics. *Studies in the History and Philosophy of Modern Physics*, *27*, 181–219.

Bub, J., Clifton, R., & Goldstein, S. (2000). Revised proof for the uniqueness theorem for no collapse interpretations of quantum mechanics. *Studies in the History and Philosophy of Modern Physics*, *31*, 95–98.

Caves, C. M., Fuchs, C. A., & Schack, R. (2002). Quantum probabilities as Bayesian probabilities. *Physical Review A*, *65*(2305), 1–6.

de Finetti, B. (1972). *Probability induction and statistics*. London: Wiley.

Demopoulos, W. (1976). The possibility structure of physical systems. In W. Harper, & C. A. Hooker (Eds.), *Foundations and philosophy of statistical theories in the physical sciences* (pp. 55–80). Dordrecht: Reidel.

Demopoulos, W. (2003). Elementary propositions and essentially incomplete knowledge: A framework of the interpretation of quantum mechanics. *Nous*, forthcoming.

Einstein, A., Rosen, N., & Podolsky, B. (1935). Can quantum-mechanical description of physical reality be considered complete? *Physical Review*, *47*, 777–780.

Finkelstein, D. (1962). The logic of quantum physics. *Transactions of the New York Academy of Sciences*, *25*, 621–637.

Fuchs, C.A. (2001). Quantum mechanics as quantum information (and only a little more) quant-ph/0205039.

Fuchs, C. A., & Peres, A. (2000). Quantum theory needs no interpretation. *Physics Today*, *53*, 70–71.

Gleason, A. M. (1957). Measures on the closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics*, *6*, 885–893.

Greenberger, D. M., Horne, M. A., & Zeilinger, A. (1989). Going beyond Bell's theorem. In M. Kafatos (Ed.), *Bell's theorem quantum theory and conceptions of the universe* (pp. 69–72). Dordrecht: Kluwer.

Hardy, L. (2001). Quantum theory from five reasonable axioms, quant-ph/0101012.

Kernaghan, M. (1994). Bell–Kochen–Specker theorem with 20 vectors. *Journal of Physics A*, *27*, L829.

Kochen, S., & Specker, E. P. (1967). The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics*, *17*, 59–87.

[7] See Demopoulos (2003), for an attempt at such an explanation.

Mermin, N. D. (1990). Simple unified form for the major no-hidden-variables theorems. *Physical Review Letters*, *65*, 3373–3376.

Pagonis, C., & Clifton, R. (1995). Unremarkable contextualism: Dispositions in Bohm's theory. *Foundations of Physics*, *25*, 281–296.

Pitowsky, I. (1989). *Quantum probability — quantum logic. Lecture Notes in Physics 321*. Berlin: Springer.

Pitowsky, I. (1994). George Boole's "conditions of possible experience" and the quantum puzzle. *British Journal for the Philosophy of Science*, *45*, 95–125.

Pitowsky, I. (1998). Infinite and finite Gleason's theorems and the logic of uncertainty. *Journal of Mathematical Physics*, *39*, 218–228.

Pitowsky, I. (2002). Range theorems for quantum probability and entanglement. In A. Khrennikov (Ed.), *Quantum theory: Reconsideration of foundations* (pp. 299–308). Tokyo: World Scientific.

Pitowksy, I., & Svozil, K. (2001). New optimal tests of quantum non-locality. *Physical Review A*, *64*, 4102–4106.

Putnam, H. (1965) Philosophy of physics. In *Mathematics, matter and method-Philosophical Papers*, Vol. 1. (pp. 79–92) Cambridge: Cambridge University Press.

Putnam, H. (1968). The logic of quantum mechanics. In *Mathematics*, *matter and method-Philosophical papers*, Vol. 1. (pp. 174–197). Cambridge: Cambridge University Press.

Ramsey, F. P. (1926). Truth and probability. In D. H. Mellor (Ed.), *F.P. Ramsey: Philosophical papers* (pp. 54–194). Cambridge: Cambridge University Press.

Savage, L. J. (1954). *The foundations of statistics*. London: Wiley.

Schack, R., Brun, T. A., & Caves, C. M. (2001). Quantum Bayes rule. *Physical Review A*, *64*, 3051–3054.

Svozil, K. (1998). *Quantum logic*. Singapore: Springer.

Vallentini, A. (1996). *Pilot-wave theory of physics and cosmology*. Cambridge: Cambridge University Press.

von Neumann, J. (1955). *Mathematical foundations of quantum mechanics*. Princeton: Princeton University Press (German edition 1932).