

On the geometry of quantum correlations

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Abstract

Consider the set \mathcal{Q} of quantum correlation vectors for two observers, each with two possible binary measurements. Quadric (hyperbolic) inequalities which are satisfied by every $q \in \mathcal{Q}$ are proved, and equality holds on a two dimensional manifold consisting of the local boxes, and all the quantum correlation vectors that maximally violate the Clauser, Horne, Shimony, and Holt (CHSH) inequality. The quadric inequalities are tightly related to CHSH, they are their iterated versions (equation 20). Consequently, it is proved that \mathcal{Q} is contained in a hyperbolic cube whose axes lie along the non-local (Popescu, Rohrlich) boxes. As an application, a tight constraint on the rate of local boxes that must be present in every quantum correlation is derived. The inequalities allow testing the validity of quantum mechanics on the basis of data available from experiments which test the violation of CHSH. It is noted how these results can be generalized to the case of n sites, each with two possible binary measurements.

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I. INTRODUCTION

The non-local character of quantum correlations is manifested by the violation of Bell inequality [1], and more generally the Clauser, Horne, Shimony, and Holt (CHSH) inequality [2]. This property has become one of the cornerstones of quantum information theory; beginning with Ekert's observation [3] that the violation of CHSH can be applied to protect the security of key distribution, the number of publications on this subject is growing at a fast rate. Still it is not completely clear why should quantum correlations violate locality the way they do.

A fresh perspective on this problem was added by Popescu and Rohrlich [4]. They demonstrated that there are non-local correlations that do not allow superluminal signaling, but nevertheless violate CHSH more strongly than any quantum correlations (and therefore cannot be realized as far as present day physics is concerned). The extreme form of these correlations became known as PR-boxes. Despite their fictitious nature they shed new light on some information theoretic problems. Thus, for example, quantum correlations sometimes provide exponential gain in communication complexity over classical correlations [5], while the availability of PR-boxes trivializes communication complexity entirely [6], [7]. Now we can ask a complementary question: why is it that quantum correlations do not extend all the way to the PR-boxes?

The relations between local correlations, quantum correlations, and the PR-boxes have a geometric representation. Imagine a source of pairs of particles, one goes to Alice and the other to Bob. Both Alice and Bob are equipped with communication boxes, each box has two settings which will be denoted by the index $i = 1, 2$ for Alice, and $j = 1, 2$ for Bob. In each run a pair of particles is sent from the source, and Alice and Bob freely choose their settings i and j . When the particles arrive to the boxes an outcome is registered in each box, which is either $+1$ or -1 . Let $s_{ij} = \pm 1$ be the product of Alice's outcome and Bob's outcome. Repeat the runs many times for the setting ij , denote the average by p_{ij} , and repeat this for all four settings. The vector $p = (p_{11}, p_{12}, p_{21}, p_{22})$ is called correlation vector.

The *local polytope* \mathcal{L} is defined to be the subset in \mathbb{R}^4 of all correlation vectors such that $p_{ij} = E(X_i Y_j)$, where X_i, Y_j are real random variables on an arbitrary probability space (Λ, Σ, μ) having values ± 1 , and $E(X_i Y_j) = \int X_i(\lambda) Y_j(\lambda) d\mu(\lambda)$ are the expectations. \mathcal{L} is the convex hull in \mathbb{R}^4 of the eight vertices,

$$\begin{aligned}
l_1 &= (1, 1, 1, 1) & l_2 &= (1, 1, -1, -1) & l_3 &= (1, -1, 1, -1) & l_4 &= (1, -1, -1, 1) \\
-l_1 &= (-1, -1, -1, -1) & -l_2 &= (-1, -1, 1, 1) & -l_3 &= (-1, 1, -1, 1) & -l_4 &= (-1, 1, 1, -1)
\end{aligned} \tag{1}$$

The facet inequalities are the eight trivial inequalities,

$$-1 \leq p_{ij} \leq 1 \quad i, j = 1, 2, \tag{2}$$

and the eight Clauser, Horne, Shimony, Holt (CHSH) inequalities [8], [9],

$$-1 \leq \frac{1}{2}p_{11} + \frac{1}{2}p_{12} + \frac{1}{2}p_{21} + \frac{1}{2}p_{22} - p_{ij} \leq 1 \quad i, j = 1, 2. \tag{3}$$

The *Popescu Rohrlich polytope* \mathcal{P} [10] is obtained by adding eight more vertices, the PR-boxes, to those in (1),

$$\begin{aligned}
n_1 &= (-1, 1, 1, 1) & n_2 &= (1, -1, 1, 1) & n_3 &= (1, 1, -1, 1) & n_4 &= (1, 1, 1, -1) \\
-n_1 &= (1, -1, -1, -1) & -n_2 &= (-1, 1, -1, -1) & -n_3 &= (-1, -1, 1, -1) & -n_4 &= (-1, -1, -1, 1)
\end{aligned} \tag{4}$$

and the inequalities for \mathcal{P} are just the trivial inequalities in (2).

As mentioned above, the vertices of \mathcal{P} can be associated with the the product of outputs of (real or hypothetical) communication boxes. The eight vertices of \mathcal{L} correspond to *local boxes* that can easily be realized. To see that think about the source as emitting pairs of balls such that the two balls in each pair are of the same color, and the colors are randomly distributed so that 50% of the pairs are red and 50% black. We assume that at the outset, before any experiment is run, Alice and Bob agree on the random variables X_i and Y_j , but afterwards they have no communication between them. Suppose that Alice's first setting, $i = 1$, is " $X_1 = 1$ if the ball is red, and $X_1 = -1$ if it is black". To realize the vertex $-l_1 = (-1, -1, -1, -1)$ Alice choose $X_2 = X_1$, and Bob chooses $Y_1 = Y_2 = -X_1$, and in this case the outputs on both sides are perfectly anti-correlated. To realize the vertex $l_4 = (1, -1, -1, 1)$ choose $Y_1 = X_1$ and $X_2 = Y_2 = -X_1$, in which case Alice and Bob outputs are perfectly correlated in the second and third experiment, and perfectly anti-correlated in the others.

The PR-boxes (4) cannot be realized in a similar manner, as far as present day physics is concerned. Take for example the vertex $n_4 = (1, 1, 1, -1)$. For the first three set-ups $i, j = 1, 1$, or $1, 2$, or $2, 1$ Alice and Bob observe balls of the same color, and in the last

setting $i, j = 2, 2$ they detect different colors. There are no classical local random variables X_i, Y_j with the above properties, which can be chosen in advance to yield these outcomes, nor are they quantum states and measurements capable of producing it. However, all the boxes in \mathcal{P} , real or imaginary, satisfy the important physical restriction of *no signaling*. This means that Bob cannot signal to Alice by changing his setting, say from $j = 1$ to $j = 2$. In the above example all Alice detects are 50% red balls and 50% black balls, no matter what Bob is doing, and the same applies to Alice.

The outputs of quantum mechanical experiments lie in between the two polytopes, there are quantum correlation vectors $q = (q_{11}, q_{12}, q_{21}, q_{22})$ such that $q \in \mathcal{P} \setminus \mathcal{L}$. Let ρ be any quantum state defined on the tensor product of two Hilbert spaces, $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$. Suppose $A_i, i = 1, 2$, are Hermitian operators on \mathbb{H}_1 , and $B_j, j = 1, 2$, on \mathbb{H}_2 , such that their spectrum is in the interval $[-1, 1]$. The general quantum correlation vector has the form,

$$q_{ij} = \text{tr}(\rho(A_i \otimes B_j)). \quad (5)$$

Tsirelson [11] proved that without loss of generality we can assume that $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{C}^2$, where \mathbb{C} is the complex field; and for four directions (unit vectors) in physical space $\mathbf{u}_i, \mathbf{v}_j$, $i, j = 1, 2$, we can set $A_i = \sigma_{\mathbf{u}_i}$, and $B_j = \sigma_{\mathbf{v}_j}$, where the σ 's are the spin operators in the corresponding directions. With this representation consider a source of pairs in the state ρ that emits the particles towards Alice and Bob. For each run of the experiment Alice can choose to measure either $\sigma_{\mathbf{u}_1}$ or $\sigma_{\mathbf{u}_2}$ with possible outcomes ± 1 , and Bob can choose between $\sigma_{\mathbf{v}_1}$ and $\sigma_{\mathbf{v}_2}$. The correlation vector is then given by $q_{ij} = \text{tr}(\rho(\sigma_{\mathbf{u}_i} \otimes \sigma_{\mathbf{v}_j}))$.

Denote by \mathcal{Q} the set of all vectors $q \in \mathbb{R}^4$ that have this form, as we vary ρ and the directions $\mathbf{u}_i, \mathbf{v}_j$. The body \mathcal{Q} is convex and satisfies $\mathcal{L} \subsetneq \mathcal{Q} \subsetneq \mathcal{P}$. Its structure has been described by Tsirelson [11], and subsequently in different equivalent ways [12], [13], [14], [15] the latter is the most compact representation given by the inequalities

$$|q_{11}q_{12} - q_{21}q_{22}| \leq \sqrt{1 - q_{11}^2} \sqrt{1 - q_{12}^2} + \sqrt{1 - q_{21}^2} \sqrt{1 - q_{22}^2}. \quad (6)$$

The boundary $\partial\mathcal{Q}$ is a complicated 3-dimensional algebraic manifold. This mathematical description has been known for a while but its physical significance is little understood. The purpose of this paper is to further advance the analysis of the structure of \mathcal{Q} . Mathematically I will demonstrate that \mathcal{Q} is contained in a 4-dimensional hyperbolic cube, whose axes lie along the PR-boxes, and whose boundary is given by quadric inequalities which are directly

related to the CHSH inequalities (3), in fact they are iterated versions of CHSH (see 20 below, other quadric inequalities satisfied by all $q \in Q$ have been previously derived in [16]). Moreover, the intersection of the boundary of the hyperbolic cube with ∂Q is a 2-dimensional sub-manifold of ∂Q corresponding to maximal quantum violations of CHSH, as explained in theorem 1 below. The physical consequences are examined subsequently, and include a calculation of the rate of local boxes that must be present in every quantum correlation vector.

II. MATHEMATICAL RESULTS

The first thing to notice is that \mathcal{P} is just the 4-dimensional unit cube, and \mathcal{L} is the 4-dimensional octahedron, so that they are polar (dual) to each other. However, while \mathcal{P} is presented in its canonical form, the 4-octahedron \mathcal{L} is rotated from its canonical representation, which is just the convex hull of

$$\begin{aligned} e_1 &= (1, 0, 0, 0) & e_2 &= (0, 1, 0, 0) & e_3 &= (0, 0, 1, 0) & e_4 &= (0, 0, 0, 1) \\ -e_1 &= (-1, 0, 0, 0) & -e_2 &= (0, -1, 0, 0) & -e_3 &= (0, 0, -1, 0) & -e_4 &= (0, 0, 0, -1) \end{aligned} \quad (7)$$

The matrix that transforms the vertices of \mathcal{L} in (1) to the respective vertices of the canonical form in (7) is

$$H = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad (8)$$

so that $2H$ is an orthogonal self adjoint Hadamard matrix. We shall denote the 4-octahedron in the canonical form by $H\mathcal{L}$. The facet inequalities of $H\mathcal{L}$ have a particularly simple form. If $r = (r_{11}, r_{12}, r_{21}, r_{22}) \in H\mathcal{L}$ then the facet inequalities are

$$\sum_{i,j=1,2} |r_{ij}| \leq 1. \quad (9)$$

Denote by $\partial\mathcal{L}_{ij}$ the facet of \mathcal{L} that corresponds to an equality on the right hand side of (3) for i, j . For example for $i = j = 2$,

$$\partial\mathcal{L}_{22} = co\{l_1, l_2, l_3, -l_4\}, \quad (10)$$

Where co stands for the convex hull. This facet is transformed by H to,

$$H(\partial\mathcal{L}_{22}) = co\{e_1, e_2, e_3, -e_4\}. \quad (11)$$

In general, any non trivial facet (3) of \mathcal{L} is transformed by H to a convex hull of four vertices with an odd number of negated e_i 's, and every trivial facet (2) moves by H to the convex hull of an even number (including zero) of negated e_i 's.

Another important feature is that all the PR-boxes in (4) are eigenvectors of $2H$, with $\pm n_1$ corresponding to the eigenvalue -1 , and the others corresponding to the eigenvalue $+1$. Also, all PR-boxes are either opposite each other or orthogonal to each other in \mathbb{R}^4 . Hence the quadric form,

$$q^t H q = \frac{1}{4}(q_{11} + q_{12} + q_{21} - q_{22})^2 + (q_{11}q_{22} - q_{12}q_{21}), \quad (12)$$

defines "Minkowskian metric" in \mathbb{R}^4 , with the axis along $\pm n_1$ playing the role of "time", and the other PR-boxes the "space" axes. The surface $q^t H q = 1$ is thus a hyperboloid. We have

Theorem 1 *If $q \in \mathcal{Q}$ then $q^t H q \leq 1$, and equality holds on a two dimensional submanifold of $\partial\mathcal{Q}$ which includes all the local boxes, and the a subset of $\partial\mathcal{Q}$ which maximally violate the CHSH inequality. We also have $q^t H q \geq -1$ for all $q \in \mathcal{Q}$.*

Proof. We shall use the following characterization due to Tsirelson [11]. If $q \in \mathcal{Q}$ there are unit vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^4$ such that $q_{ij} = \mathbf{x}_i \cdot \mathbf{y}_j$ for $i, j = 1, 2$. Moreover, if $q \in \partial\mathcal{Q}$ then the $\mathbf{x}_i, \mathbf{y}_j$'s are in the same plane. Put

$$\mathbf{a} = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2), \quad \mathbf{a}^\perp = \frac{1}{2}(\mathbf{x}_1 - \mathbf{x}_2), \quad \mathbf{b} = \frac{1}{2}(\mathbf{y}_1 + \mathbf{y}_2), \quad \mathbf{b}^\perp = \frac{1}{2}(\mathbf{y}_1 - \mathbf{y}_2), \quad (13)$$

then \mathbf{a} and \mathbf{a}^\perp are orthogonal to each other, \mathbf{b} and \mathbf{b}^\perp are orthogonal, $\|\mathbf{a}\|^2 + \|\mathbf{a}^\perp\|^2 = 1$ and $\|\mathbf{b}\|^2 + \|\mathbf{b}^\perp\|^2 = 1$ where $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^4 . Then a straightforward calculation shows

$$(Hq)_{11} = \mathbf{a} \cdot \mathbf{b}, \quad (Hq)_{12} = \mathbf{a}^\perp \cdot \mathbf{b}, \quad (Hq)_{21} = \mathbf{a} \cdot \mathbf{b}^\perp, \quad (Hq)_{22} = \mathbf{a}^\perp \cdot \mathbf{b}^\perp. \quad (14)$$

Put $\|\mathbf{a}\| = \cos \alpha$, and $\|\mathbf{b}\| = \cos \beta$. Assume that $q \in \partial\mathcal{Q}$ and the $\mathbf{x}_i, \mathbf{y}_j$'s are in the same plane and let θ be the angle between \mathbf{a} and \mathbf{b} . If q is in the part of $\partial\mathcal{Q}$ just above the facet

$\partial\mathcal{L}_{22}$ in (10), we deduce from (11) that $(Hq)_{11}, (Hq)_{12}, (Hq)_{21} \geq 0$ and $(Hq)_{22} \leq 0$. Using (9) we can calculate the value of the CHSH expression,

$$CHSH = \sum_{i,j=1,2} |(Hq)_{ij}| = \cos(\alpha - \beta) \cos \theta + \sin(\alpha + \beta) \sin \theta. \quad (15)$$

Suppose that we have fixed the lengths $\|\mathbf{a}\| = \cos \alpha$, and $\|\mathbf{b}\| = \cos \beta$, then the maximum on the right hand side of (15) is obtained for θ which satisfies

$$\tan \theta = \frac{\sin(\alpha + \beta)}{\cos(\alpha - \beta)}. \quad (16)$$

The value of the CHSH for this choice is,

$$\max CHSH = \max_{\theta} \sum_{i,j=1,2} |(Hq)_{ij}| = \sqrt{\cos^2(\alpha - \beta) + \sin^2(\alpha + \beta)}, \quad (17)$$

with the absolute maximum $\sqrt{2}$ (the Tsirelson bound) obtained when we take $\alpha = \beta = \frac{\pi}{4}$, (and $\theta = \frac{\pi}{4}$).

The matrix $2H$ is both self adjoint and orthogonal and therefore we have $H^2 = \frac{1}{4}I$. Substituting the values from (14) to (12) we get,

$$q^t H q = 4(Hq)^t H(Hq) = [\cos(\alpha - \beta) \cos \theta + \sin(\alpha + \beta) \sin \theta]^2 - \sin(2\alpha) \sin(2\beta). \quad (18)$$

Again, suppose that the lengths $\|\mathbf{a}\| = \cos \alpha$, and $\|\mathbf{b}\| = \cos \beta$ are fixed, then for $q \in \partial\mathcal{Q}$ above the facet $\partial\mathcal{L}_{22}$ the maximum value of $q^t H q$ is obtained at θ in (16) and it is,

$$\max(q^t H q) = \cos^2(\alpha - \beta) + \sin^2(\alpha + \beta) - \sin(2\alpha) \sin(2\beta) = 1 \quad (19)$$

It is straightforward to check that $q^t H q = 1$ for all the local boxes in (1). The quantum correlation vectors at which we obtain the absolute extrema of CHSH are $\pm \frac{1}{\sqrt{2}}n_k$, where the $\pm n_k$ are the PR-boxes (4). Recall that the PR-boxes n_2, n_3, n_4 in (4) are also eigenvectors of $2H$ with eigenvalue $+1$ and therefore we have $(\frac{1}{\sqrt{2}}n_k)^t H(\frac{1}{\sqrt{2}}n_k) = 1$ for $k = 2, 3, 4$. Hence, the above argument can be repeated with regard to the part of $\partial\mathcal{Q}$ above $\partial\mathcal{L}_{12}$ and above $\partial\mathcal{L}_{21}$. Since n_1 is an eigenvector of $2H$ with eigenvalue -1 we have $(\frac{1}{\sqrt{2}}n_1)^t H(\frac{1}{\sqrt{2}}n_1) = -1$, and the correlation vector $\frac{1}{\sqrt{2}}n_1$ does not lie on the surface of the hyperboloid $q^t H q = 1$, nor does the part of $\partial\mathcal{Q}$ above $\partial\mathcal{L}_{11}$; however we have $q^t H q \geq -1$ for all $q \in \mathcal{Q}$. ■

Corollary 2 *The iterated CHSH: for all $q \in \mathcal{Q}$ we have*

$$\begin{aligned} -1 \leq & -\frac{1}{8}(-q_{11} + q_{12} + q_{21} + q_{22})^2 + \frac{1}{8}(q_{11} - q_{12} + q_{21} + q_{22})^2 + \\ & \frac{1}{8}(q_{11} + q_{12} - q_{21} + q_{22})^2 + \frac{1}{8}(q_{11} + q_{12} + q_{21} - q_{22})^2 \leq 1. \end{aligned} \quad (20)$$

and, by symmetry, another three inequalities of the same form, each with one of the components of (20) having a minus sign.

Proof. Since the n_k 's are orthogonal in pairs, we can write each correlation vector $q = (q_{11}, q_{12}, q_{21}, q_{22})$ in terms of the orthogonal basis $\{n_i\}$. Since $Hn_1 = -\frac{1}{2}n_1$ and $Hn_k = \frac{1}{2}n_k$ for $k = 2, 3, 4$, this yields $q^t H q = -\frac{1}{8}(n_1^t q)^2 + \frac{1}{8}(n_2^t q)^2 + \frac{1}{8}(n_3^t q)^2 + \frac{1}{8}(n_4^t q)^2$, and from theorem 1 we get (20). From symmetry it is obvious that we can choose any of the PR-boxes $\pm n_k$ to play the role of the "time" (eigenvalue = -1) axis, and the other three the "space" axes, simply by replacing the Hadamard matrix $2H$ by another. In this way we can get four hyperboloids and \mathcal{Q} is contained in their intersection, each yields another inequality of the form (20). ■

III. PHYSICAL CONSEQUENCES

We can easily derive the experimental arrangements which will give rise to the extrema (17, 19). Using the fact that $H^2 = \frac{1}{4}I$ we can invert the relations in (14), and represent q_{ij} in terms of the parameters α, β, θ

$$q_{11} = \cos(\alpha + \beta - \theta), \quad q_{12} = \cos(\alpha - \beta - \theta), \quad q_{21} = \cos(\alpha - \beta + \theta), \quad q_{22} = \cos(\alpha + \beta + \theta) \quad (21)$$

with θ given by (16). From these values the angles between the directions \mathbf{u}_i and \mathbf{v}_j in the measurement of $\sigma_{\mathbf{u}_i} \otimes \sigma_{\mathbf{v}_j}$ can be derived.

More generally, we can formulate the iterated CHSH in term of the observables A_i and B_j in (5), denote for $i, j = 1, 2$,

$$C_{ij} = \frac{1}{2}A_1 \otimes B_1 + \frac{1}{2}A_1 \otimes B_2 + \frac{1}{2}A_2 \otimes B_1 + \frac{1}{2}A_2 \otimes B_2 - A_i \otimes B_j. \quad (22)$$

Then the general iterated CHSH reads

$$-1 \leq \frac{1}{2} |tr(\rho C_{11})|^2 + \frac{1}{2} |tr(\rho C_{12})|^2 + \frac{1}{2} |tr(\rho C_{21})|^2 + \frac{1}{2} |tr(\rho C_{22})|^2 - |tr(\rho C_{ij})|^2 \leq 1, \quad (23)$$

for $i, j = 1, 2$ and any state ρ on $\mathbb{H}_1 \otimes \mathbb{H}_2$. Hence, the numbers $|tr(\rho C_{ij})|^2$ satisfy all the CHSH inequalities (3), however, they do not necessarily satisfy the trivial inequalities (2). We can use this inequality to test the validity of quantum mechanics, using the data that has already been collected in many experiments that test the violation of CHSH. By quantum mechanics the same data must satisfy the iterated CHSH, and the inequality is tight.

Perhaps the easiest way to grasp the interpretation these inequalities is in terms of *non-local deterministic hidden variable theories* such as Bohm's (see e.g., [17]). Given the value of the hidden variable λ (in Bohm's theory, the exact initial positions of the particles of an EPR pair) we can predict at the outset the outcomes of each of the four measurements of $\sigma_{\mathbf{u}_i} \otimes \sigma_{\mathbf{v}_j}$, $i, j = 1, 2$. Suppose that we want to recover the quantum correlation vector $q = (q_{11}, q_{12}, q_{21}, q_{22})$ that violates the CHSH inequality. We sample at random the hidden variables λ according to the measure μ on the space of hidden variables (in Bohm's theory, initial values of the positions of the particles according to the distribution $|\psi|^2$ at time 0, where ψ is the full quantum state). For each value of the hidden variable we calculate the deterministic outcomes of all four experiments, the result is a ± 1 four-dimensional vector. Finally, to get q , we take the average of the vectors. To obtain a result that violates CHSH some of the ± 1 vectors in the sample must be PR-boxes, but how many? In other words, what is the minimal frequency with which a PR-box should appear in the hidden variable sample that yields the correlation vector q ? (A similar problem is considered in [18], [19]).

Assume that $q \in Q$ is above the facet $\partial\mathcal{L}_{22}$ of \mathcal{L} , given in (10). In this case we can represent q as a convex combination

$$q = \eta_1 l_1 + \eta_2 l_2 + \eta_3 l_3 - \eta_4 l_4 + \eta n_4, \quad \eta_i, \eta \geq 0, \quad \eta + \sum \eta_i = 1. \quad (24)$$

The l 's are the local boxes in $\partial\mathcal{L}_{22}$ and n_4 is the PR-box above $\partial\mathcal{L}_{22}$. Calculating η , the coefficient of the PR box, we get,

$$\eta = \frac{1}{2}(q_{11} + q_{12} + q_{21} - q_{22}) - 1 \leq \sqrt{2} - 1, \quad (25)$$

and this is the minimal rate of the PR-box n_4 in the average (24). This result has an information theoretic formulation: Suppose that Alice and Bob prepare a key using BB84, then $\eta = p_{NL}$ is the minimal rate with which Eve should prepare and send a PR-box if she is to deceive Alice and Bob that nobody listens on their line [20]. Somewhat more mysteriously it is also related to the critical security criteria of BB84 against symmetric individual attacks [21].

We can also formulate the limitation on quantum correlations in terms of the coefficients η_i in (24). Again, if we consider $q \in Q$ above the facet $\partial\mathcal{L}_{22}$, the iterated CHSH inequality in (12, 20, 23) is equivalent to the formula

$$\eta_1 + \eta_2 + \eta_3 + \eta_4 \geq 1 - 2\sqrt{\eta_1\eta_4 + \eta_2\eta_3}, \quad (26)$$

with equality on the set of maximally violating quantum correlations described in the proof of theorem 1. The interesting aspect about this inequality is that it involves only the rates of the classical local boxes in our hypothetical ensemble. This inequality bounds the rates of local boxes that must be present in any quantum correlation vector q . In the symmetric case when all the η_i 's are equal, $\eta_i = \eta_0$ we have

$$\eta_0 \geq \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}} \right). \quad (27)$$

and the total frequency of classical boxes that should be used to recover q is $4\eta_0 \geq 2 - \sqrt{2}$. The number on the right in (27) is also the critical value of the quantum bit error rate above which BB84 becomes insecure against individual symmetric attacks [21].

It seems that these results can be readily generalized to the case of n particles, and two binary traceless measurements on each. Werner and Wolf [13] established that the local correlation vectors (of dimension 2^n) form a polytope, with 2^{2^n} facet inequalities, all generalizations of CHSH. The polytope is a 2^n -dimensional octahedron. A Hadamard matrix (with a suitable normalization) will transform this polytope to its canonical position relative to its polar, the unit 2^n -dimensional cube. The Tsirelson boundary also has a detailed description in this case, and it seems to me that the formulation and proof of theorem 1 can be repeated.

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